MATHEMATICAL ANALYSIS OF SOME NON-LINEAR PROGRAMMING PROBLEMS

A THESIS

Submitted in Partial Fulfilment of the Requirements for
THE DEGREE OF DOCTOR OF PHILOSOPHY

By

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CERTIFICATE

This is to certify that the thesis entitled 'Mathematical Analysis of some Non-Linear Programming Problems' by Chhajju Ram Beetor, for the award of the Degree of Doctor of Philosophy, of the Indian Institute of Technology, Kanpur, is a record of bonafide research work carried out by him under my supervision and guidance for the last three years. The thesis has, in my opinion reached the standard fulfilling the requirements for the Doctor of Philosophy Degree. The results embodied in this thesis have not been submitted to any other University or Institute for the award of any degree or diploma.

J. N. Kapur

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July - 1968

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Chhajju Ram Bector

Chhajju Ram Bector

TO

MY FATHER

AND

MY WIFE

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LIST OF SYMBOLS

R	The set of real numbers
R^n	n-Euclidean Space
$\forall$	For all
$\exists$	There exists
$\in$	Is a member of or belonging to or belongs to
ε	Epslin
$f \circ g$	f Composition g (composition of functions)
$\cup$, $\cap$	Union symbol for sets, Intersection symbol for sets
e_j	unit vector with j-th component as unity
C^p	Class of continuous functions whose p-th order partial derivatives exist and are continuous
$\left.\begin{matrix}\varphi, \psi, f, g \\ f_i, g_i \text{ etc.}\end{matrix}\right\}$	Continuous functions
$\{'\}$	Prime
∇_x	Gradient Operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ----, \frac{\partial}{\partial x_n}\right)'$
∇_λ	Gradient Operator $\left(\frac{\partial}{\partial \lambda_1}, \frac{\partial}{\partial \lambda_2}, ----, \frac{\partial}{\partial \lambda_m}\right)'$
$\left.\begin{matrix}\text{Max}(\varphi(x); x \in P) \\ \max\limits_{x \in P} \varphi(x)\end{matrix}\right\}$	Max $\varphi(X)$ for $X \in P$
$\Rightarrow$, $\Leftrightarrow$	This implies that, This is equivalent to
$\subset$, $\supset$	Is a subset of, Is a superset of

LIST OF ABBREVIATIONS

CX(CV)	Convex (Concave)
SX(SV)	Strictly Convex (Strictly Concave)
SPCX(SPCV)	Strong Pseudo-Convex (Strong Pseudo-Concave)
PCX(PCV)	Pseudo-Convex (Pseudo-Concave)
SQX(SQV)	Strictly Quasi-Convex (Strictly Quasi-Concave)
EQX(EQV)	Explicit Quasi-Convex (Explicit Quasi-Concave)
QX(QV)	Quasi-Convex (Quasi-Concave)
CXL	Convex Like
WCXL(WCVL)	Weakly Convex Like (Weakly Concave-Like)
WSPCXL(WSPCVL)	Weakly Strong Pseudo-Convex Like (Weakly Strong Pseudo-Concave Like)
QXL(QVL)	Quasi-Convex Like (Quasi-Concave Like)
N.L.I.F.P.(P.)	Non-Linear Indefinite Functional Programming (Problem)
L.F.F.P.(P.)	Linear Fractional Functional Programming (Problem)
N.L.F.F.P.(P.)	Non-Linear Fractional Functional Programming (Problem)
SPV P.(P.)	Strong Pseudo-Concave Programming (Problem)
SPX P.(P.)	Strong Pseudo-Convex Programming(Problem)
PCV P.(P.) or PV P.(P.)	Pseudo-Concave Programming (Problem)
PCX P.(P.) or PX P.(P.)	Pseudo-Convex Programming (Problem)
EX P.(P.)	Explicit Quasi-Convex Programming (Problem)
EV P.(P.)	Explicit Quasi-Concave Programming (Problem).

SYNOPSIS

"MATHEMATICAL ANALYSIS OF SOME NON-LINEAR PROGRAMMING [illegible]" a thesis submitted in partial fulfilment for the Degree of Doctor of Philosophy, by Chhajju Ram Seetor M.A. to the Department of Mathematics, Indian Institute of Technology, Kanpur, July, 1968.

The present thesis deals with some theoretical and computational aspects of, 'Non-Linear Indefinite Functional Programming'; 'Non-Linear Fractional Functional Programming'; problem of optimising an explicit quasi-convex (quasi-convex) or an explicit quasi-concave (quasi-concave), function, and the problem of optimizing a Strong Pseudo-Concave (Pseudo-Concave) function, over a convex region.

Chapter I deals with the introduction with a brief history and review of the subject of mathematical programming.

The purpose of Chapter II is to deal with a variety of topics related with certain theoretical aspects of fundamental nature for the problem of 'Non-Linear Indefinite Functional Programming' in which the objective function, which is neither convex nor concave, but is more general than a concave function, is the product of two non-linear strictly positive (assumed to be differentiable if and when necessary) concave functions, and the constraint set is a convex set (assumed to be constrained by quasi-convex functions if and when needed). This problem is a generalization of the problem of indefinite quadratic programming considered by Kanti Swarup (1966,1967) in which the product

of two strictly positive linear functions is to be optimised over a convex polyhedral set (a convex set constrained by convex functions). Such problems arise in the situations of pure competetions in a market. The main results established in this section are analogous to those established by Hadley (1964) for a convex programming problem and that the Kuhn-Tucker differential conditions are necessary and sufficient for the existence of an optimal solution of such a problem. The Converse Duality Theorem of Mangasarian (1962) and Huard (1962,1963) has also been extended.

Chapter III is mainly concerned with 'Non-linear Fractional Functional Programming' in which, the objective function, which is neither convex nor concave but is more general than a convex (concave) function is the ratio of a non-negative convex (concave) function f to a strictly positive concave (convex) function g (if the function g be linear the non-negativity restriction) on f can be omitted and the constraint set is a convex set. If necessary the functions f,g are assumed to be differentiable and the constraint set to be constrained by quasi-convex functions. The problem is an extension of the problem of linear fractional functional programming considered by Isbell and Marlow (1956), Charnes and Cooper (1962), Dorn (1962), Bela Martos (1960) and Gilmore and Gomory (1963) Kanti Swarup (1965). The main results established are analogous to those established for non-linear indefinite functional programming considered in chapter III. The two most interesting properties that for a linear fractional functional programming problem, (i) a local optimum is a global optimum,

(ii) the optimum, if it occurs at a finite point, also occurs at an extreme point of the constraint set, follow as a natural consequence of the main results proved in this chapter.

Recently Mangasarian (1965), Bela Martos (1965) introduced respectively the notions of pseudo-convex (concave) functions and explicit quasi-convex (concave) functions. In Chapter IV results of Hadley (1964) for convex programming have been extended to the optimization of an explicit quasi-convex function, over a convex set. This also includes the extension of the result established by Bela Martos (1965), that 'every local minimum of an explicit quasi-convex function over a convex polyhedral set is a global minimum also', to that 'every local minimum of an explicit quasi-convex function over a convex set is a global minimum also'. Further in this chapter a new concept of Strong Pseudo-Convex functions, which belong to the class of functions intermediate between class of differentiable convex functions and class of pseudo-convex functions, is introduced and some of their properties are investigated. Another important result proved is that local minimum of a pseudo-convex function, when minimised over a convex set, is global minimum also. Mangasarian (1965) proved this result by using the fact that every pseudo-convex function is strictly quasi-convex also. Here, however, a direct proof, without using this property is established. Nature of products, quotients, rational powers and composition of convex like functions is established. A few results of Berge (1963), Berge and Houri (1965) and Arrow and Enthoven (1961) for quasi-convex

functions, are proved for explicit quasi-convex, pseudo-convex and strong pseudo-convex functions. By doing so it has been possible to characterize indefinite functional programs, fractional functional programs, and composite functional programs in which the objective function is the function of convex (concave) functions, as explicit quasi-convex (concave) programs, pseudo-convex (concave) programs and strong pseudo-convex (concave) programs. Another interesting result of this chapter is that the ratio of the square of a non-negative convex function f to a strictly positive concave function g (if both f and g be linear then non-negativity restriction on f may be omitted) is a convex function. This result is applied to certain non-convex mathematical programming problems to reduce them to convex programming problems.

Finally in Chapter V, a computational technique similar to 'method of feasible directions' given by Zoutendijk (1959) is developed for obtaining the solution of a Strong Pseudo-Concave (Pseudo-concave) Programming problem in which the objective function to be maximized is a differentiable Strong Pseudo-convex (Pseudo-convex) function and the constraint set is constrained by differentiable quasi-convex and linear functions subjected to regularity condition given by Abadie (1967). Further a finite iteration 'adjacent vertex method' similar to that provided by Beale (1959) and Kanti Swarup (1965) is provided for obtaining the global optimum of a special type of non-linear fractional functional program in which the objective function is the sum of the odd powers of a linear fractional function with denominator strictly positive over a convex set constrained by linear functions.

The work included in the thesis is based on the following research papers written by the author.

1. "Non-Linear Indefinite Functional Programming with Non-Linear Constraints," Cahiers du Centre d'Etude de Rech. Oper., Vol. 9, No. 4, 1967.
2. "Certain Aspects of Duality in Non-Linear Indefinite Functional Programming," Research Report, Department of Mathematics, I.I.T.K. 1967.
3. "Non-Linear Fractional Functional Programming with Non-Linear Constraints," ZAMM (Germany), Vol. 48, No. 4, 1968.
4. "Duality in Fractional and Indefinite Programming," ZAMM (Germany) Vol. 48, No. 6, 1968.
5. "A Note on Explicit Quasi-Convex and Pseudo-Convex Functions," To be published.
6. "Strong Pseudo-Convex Functions". To be published.
7. "Nature of Quotients, Products and Rational Powers of Convex (Concave) like Functions," (with S.K. Gupta) Accepted for publication in The Mathematics Student, Jour. of Indian Mathematical Soc.
8. "Method of Feasible Directions in Strong Pseudo-Concave (Pseudo-Concave) Programming." To be published.
9. "Programming Problems with Convex Fractional Functions," Operations Research, Vol. 16, No. 2, 1968, pp. 383-391.

10. "A Special Type of Non-Linear Fractional Functional Programming Problem," To be published.

In addition the following additional papers published by the author have also been referred to in this thesis, though these are not included here.

1. "Indefinite Quadratic Programming with Standard Errors in Objective," Cahiers du Centre d'Etudes de Rech. Oper. Vol. 10, 1968.

2. "Indefinite Cubic Programming with Standard Errors in Objective Functions," Unternehmensforschung, Vol. 12, No. 2, 1968, pp. 113-120.

3. "Indefinite Quadratic Fractional Functional Programming", Accepted for publication in Metrika (Austria).

4. "Some Aspects of Non-Linear Indefinite-Fractional Functional Programming," Accepted for publication in Cahiers du Centre d'Etudes de Rech. Oper.

CHAPTER I

INTRODUCTION

The world to-day is experiencing the complex organisational problems which have significant impact on the modern society. Such problems as the most efficient manner in which to run the economy of a nation, or the optimum deployment of the armed forces so as to maximize a country's chances of winning a war, or maximising the ingredients of a fertiliser to meet agricultural specifications at a minimum cost, are some of those serious questions which every nation is facing to-day. Research on how to formulate and solve such problems led to the appearance of a new class of optimisation problems which cannot be solved by using the usual classical techniques such as those based on differential calculus or calculus of variations. Among those

is one named as the class of 'Mathematical Programming Problems' (M.P.P.), within the realm of which the work contained in the present thesis falls.

The main aim of this chapter is to give a survey of both theoretical and computational developments of past few years in non-linear programming (N.L.P.). This survey is not intended to be exhustive, but is merely representative of the totality of developments of those aspects which will enable the author to place his own contributions in their proper perspective. To make the survey more precise and understandable, the present chapter is divided into three major sections. Section I contains the preliminaries and notations to be followed very frequently not only in this chapter but, as far as possible, throughout the whole thesis. Section II gives the brief history of the subject of M.P. and also a review of these aspects of the work done in this field by other research workers which are relevant to the present thesis. Section III contains the summary of those results obtained by the author which are included in the remaining chapters of this thesis.

SECTION - I

PRELIMINARIES AND NOTATIONS

Let X be an n vector of variables, or points in R^n, a Euclidean space of dimension n. Let S be an subset in R^n and X_1, X_2 any two vectors belonging to S. Let C denote the class of all those continuous, single valued function defined on S, which are such that $f: S \longrightarrow I \subset R$, the set of reals, and let C^p, $p = 1, 2, ——$, (p being finite), denote the class of all those $f \in C$, every p-th order partial derivative of which exists and is continuous in S. Assume that the vector $X = (x_1, x_2, ——, x_n)'$

means that x_j is the j-th component of the vector $X \in S$ and similarly $X_k = (x_{k1}, x_{k2}, \text{———}, x_{kn})'$ means that x_{kj} is the j-th component of $X_k \in S$. Let ∇_x denote the gradient operator $\left(\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \text{-----}, \frac{\partial}{\partial x_n}\right)'$ and 'prime' the transpose.

Note: Unless specified otherwise, throughout the work it is assumed that each of the functions involved in problems to be considered is continuous, real, scalar and single valued. However, for convenience, at some places the continuity has been stated.

We now define certain important concepts in E^n.

(i) CONVEX SET: A set is defined to be a convex set, if for every pair of points X_1, X_2 in the set, the point $\lambda X_1 + (1-\lambda)X_2$ for all $\lambda \in [0,1]$ is in the set.

(ii) NON-CONVEX SET: A set is defined to be a non-convex set if for some points X_1, X_2 in the set, the point $\lambda X_1 + (1-\lambda)X_2$ for some $\lambda \in]0,1[$ is not in the set, i.e. a set is a non-convex set if it is not a convex set.

Remark 1. Unless otherwise specified, throughout the work to follow S will denote a convex subset and G a non-convex subset of E^n.

(iii) EXTREME POINT: A point $X \in S$ is called its extreme point, iff there do not exist other points X_1, X_2 in S, $X_1 \neq X_2$ such that

$$X_\lambda = \lambda X_1 + (1-\lambda) X_2 \quad \text{for all} \quad \lambda \in]0,1[.$$

(iv) A HYPERPLANE: The set of points X satisfying

$$c_1 x_1 + c_2 x_2 + \text{-----} + c_n x_n = Z \quad (\text{not all } c_i = 0)$$

defines a hyperplane for given values of the c_i and Z.

If we write $C = (c_1, c_2, \text{------}, c_n)'$, then the above hyperplane can be written as $Z = C'X$, whereas $C'X$ defines the inner product of two vectors C and X in E^n.

(v) CONVEX COMBINATION: A convex combination of a finite number of points $X_1, X_2, \text{------}, X_m$ in S is defined as a point

$$X = \sum_{i=1}^{m} \mu_i X_i \quad , \quad \mu_i \geq 0, \quad i = 1, 2, --- m \; ; \quad \sum_{i=1}^{m} \mu_i = 1$$

We now define a function f over the convex set S, to be

(i) LINEAR, iff for all X_1, X_2 in S, $f\,[\ell X_1 + mX_2] = \ell f(X) + mf(X_2)$, ℓ and m being real numbers, is satisfied.

(ii) CONVEX (CX), if for all X_1, X_2 in S,

$$f[\lambda X_1 + (1-\lambda) X_2] \leq \lambda f(X_1) + (1-\lambda) f(X_2) \quad \text{for all} \quad \lambda \in [0,1].$$

or equivalently, when $f \in C^1$,

$$f(X_1) - f(X_2) \geq (X_1 - X_2)' \nabla_x f(X_2)$$

(iii) STRICTLY CONVEX (SX), if for all X_1, X_2 in S, $X_1 \neq X_2$

$$f[\lambda X_1 + (1-\lambda) X_2] < \lambda f(X_1) + (1-\lambda) f(X_2) \quad \text{for all} \quad \lambda \in]0,1[.$$

(iv) PSEUDO-CONVEX (PCX), if for all X_1, $X_2 \in S$ and $f \in C^1$,

$$(X_1 - X_2)' \nabla_x f(X_2) \geq 0 \implies f(X_1) \geq f(X_2)$$

i.e. if $\quad f(X_1) < f(X_2) \implies (X_1 - X_2)' \nabla_x f(X_2) < 0$

(v) STRICTLY QUASI-CONVEX (SQX), if for every X_1 and $X_2 \in S, X_1 \neq X_2$

$$f(X_1) < f(X_2) \Rightarrow f[\lambda X_1 + (1-\lambda) X_2] < f(X_2) \quad \text{for all} \quad \lambda \in]0,1[$$

(vi) EXPLICIT QUASI-CONVEX (EQX), if for all X_1, $X_2 \in S$ satisfying $f(X_1) \neq f(X_2)$

$$f[\lambda X_1 + (1-\lambda)X_2] < \text{Max}[f(X_1), f(X_2)] \quad \text{for all } \lambda \in]0,1[.$$

(vii) QUASI-CONVEX (QX). Below we give different equivalent definitions of QX functions.

1. A function f is QX on S if for all X_1, $X_2 \in S$ and all $\lambda \in [0,1]$,

$$f[\lambda X_1 + (1-\lambda)X_2] \leq \text{Max}[f(X_1), f(X_2)]$$

or

$$f(X_1) \leq f(X_2) \Rightarrow f[\lambda X_1 + (1-\lambda)X_2] \leq f(X_2)$$

or

$$f(X_1) < f(X_2) \Rightarrow f[\lambda X_1 + (1-\lambda)X_2] \leq f(X_2)$$

2. A function $f \in C^1$ is QX on S if for all X_1, X_2 in S,

$$f(X_1) \leq f(X_2) \Rightarrow (X_1 - X_2)' \nabla_x f(X_2) \leq 0$$

i.e.

$$(X_1 - X_2)' \nabla_x f(X_2) > 0 \Rightarrow f(X_1) > f(X_2)$$

or

$$f(X_1) < f(X_2) \Rightarrow (X_1 - X_2)' \nabla_x f(X_2) \leq 0$$

i.e.

$$(X_1 - X_2)' \nabla_x f(X_2) > 0 \Rightarrow f(X_1) \geq f(X_2)$$

3. A function f is defined to be QX on S iff,

the set, $\{X ; X \in S, \ f(X) \leq \alpha\}$

or the set $\{X ; X \in S, \ f(X) < \alpha\}$

is convex for all α .

Furthermore, a function f on a convex set S is defined to be Concave (CV), Strictly Concave (SV), Pseudo-Concave (PCV), Strictly-Quasi Concave (SQV), Explicit Quasi-Concave (EQV) or Quasi-Concave (QV) according as , -f, is, CX, SCX, PCX, SQX, EQX or QX respectively

Also on the set S, f is defined to be,

(viii) PSEUDO-MONOTONIC (PCM), iff f is both PCX and PCV.

(ix) EXPLICIT QUASI-MONOTONIC (EQM), if f is both, (i) EQX and QV, or (ii) EQV and QX or (iii) EQX and EQV.

(x) QUASI-MONOTONIC (QM), iff it is both QX and QV.

Note: (i) As stated by Mangasarian [148] , the convexity of the set S above, for defining f to be a PCX (or PCV) and hence a PCM function is not essential.

(ii) In the work to follow, we will refer very often to any of the above function as a Convex-Like (CXL) function.

Mathematical Programming Problems:

M.P.P.'s in general are concerned with the efficient use or allocation of limited resources to optimise desired objectives. These problems are characterised by the large number of solutions that satisfy the basic conditions of each problem. The selection of a particular solution as the best solution to a problem depends on some aim or over-all objective that is implied in the statement of the problem.

Programming problems appear in a variety of contexts in practical problems. A. Charnes and W.W. Cooper in 1950 began to explore complex problems of many interconnected activities such as 'Production in the Industry'. Also programming problems have been encountered in the fields

of Economics, Pure Sciences, Biological Sciences and Engineering. There may exist differences in the objectives to be achieved and the quantum of efforts to be applied, it is, however, possible to abstract the underlying essential similarities in the management of those seemingly essentially different systems. For achieving this it is essential to have a look at the structure, the state of system and the objective to be fulfilled in order to construct a statement of actions to be performed, their timing and their quantity-called, a program, or, schedule, which will permit the system to move from a given position towards the definite objective. If the system exhibits a structure which can be represented by a mathematical equivalent, called a mathematical model and if the system can also be quantified then some theoretical results may be obtained for the existence of the 'best schedule of actions' and some computational methods may be obtained for choosing this 'best schedule of actions' among the 'alternatives'. Such a use of the mathematical model is termed as 'Mathematical Programming'.

Mathematical formulation of a mathematical programming problem is stated as follows:

It is desired to determine values for n variables $x_1, x_2, \text{---}, x_n$ which satisfies the m inequalities or equations

$$g_i(x_1, x_2, \text{----}, x_n) \left\{ \leqslant, =, \geqslant \right\} b_i \quad (i = 1, 2, \text{---}, m) \qquad (1.1.1)$$

and in addition optimises (maximises or minimises) the function

$$Z = \varphi(x_1, x_2, \text{----}, x_n) \qquad (1.1.2)$$

An equivalent vector notation form is as follows:

We have to find a vector $X \in E^n$ which satisfies the m inequalities

or equations

$$g_i(X)\left\{\leqslant, =, \geqslant\right\} b_i \quad , (i = 1, 2, ---, m) \qquad (1.1.2)$$

and in addition optimizes (Maximizes or Minimizes) the function

$$Z = \varphi(X) \qquad (1.1.2)$$

Before we proceed further we define the following concepts in connection with the above M.P.P.

(i) CONSTRAINTS: The restrictions (1.1.1) are called constraints.

(ii) FEASIBLE SOLUTION (F.S.): An nx1 column vector $X=(x_1, x_2, --- x_n)$, if it satisfies the constraints (1.1.1) is called a F.S.

(iii) CONSTRAINT SET OR SET OF FEASIBLE SOLUTIONS: A set P given by

$$P = \left\{X \; ; \; g_i(X)\left\{\leqslant, =, \geqslant\right\} b_i \; , \; i = 1, 2, ---, m\right\} \qquad (1.1.3)$$

is called a constraint set or a set of feasible solutions.

(iv) OBJECTIVE FUNCTION (O.F.): The function φ in (1.1.2) is called Objective Function.

(v) STRONG RELATIVE MAXIMUM: The function φ is said to take on a strong relative maximum over P at the point $X_0 \in P$ if there exists an $\epsilon > 0$ such that for every $X \neq X_0$ in an ϵ-neighbourhood of X_0 for which $X \in P$, we have $\varphi(X) < \varphi(X_0)$.

(vi) WEAK RELATIVE MAXIMUM: A point $X_0 \in P$ is said to be a weak relative maximum of φ over the set P, if it does not have a strong relative maximum at X_0 and if there exist an $\epsilon > 0$ such that for every point $X \in P$ in an ϵ-neighbourhood of X_0, we have $\varphi(X) \leqslant \varphi(X_0)$.

Note: When it is unnecessary to distinguish between strong and weak relative maxima, we shall simply refer to relative maxima.

(vii) GLOBAL MAXIMUM: A point $X_0 \in P$ is defined to a global maximum of φ over P if for all $X \in P$ we have $\varphi(X) \leqslant \varphi(X_0)$.

Note: (i) Many times the word 'Global Maxima' will be replaced by the word 'MAXIMUM' only.

(ii) The modifications necessary to define 'Strong' and 'Relative' Minima and 'Global' Minima are obvious.

(iii) For a maximization or minimisation problem, the word 'Optimal solution' will mean respectively the global maximum or global minimum.

In (1.1.1), the g_i's are assumed to be specified functions and b_i are assumed to be known constants. Furthermore, in (1.1.1), one and only one of the signs $\leqslant, =, \geqslant$ holds for each constraint but the sign may vary from one constraint to another. The values of m and n need not be related in any way, i.e. m can be allowed to be greater than, less than or equal to n. In particular m can be Zero so that we include cases where there are no constraints (1.1.1). Usually some of the variables are restricted to be non-negative. Unless otherwise specified, (1.1.1) and (1.1.2) is interpreted as a problem in which it is desired to find numerical values for the n-components $x_1, x_2, \text{---}, x_n$ of the vector X which optimise (1.1.2) subject to (1.1.1) and non-negativity requirements.

DIVISION OF M.P.P.'s: M.P.P.'s can be divided into two major classes.

1. NON-LINEAR PROGRAMMING PROBLEMS (N.L.P.P.): If in (1.1.1) and (1.1.2) either the O.F. φ, or at least one of the constraint functions g_i, or both are non-linear or the variables can take integral values only, then the M.P.P. is termed as N.L.P.P.

2. LINEAR PROGRAMMING PROBLEM (L.P.P.): If in (1.1.1) and (1.1.2) both, the O.F. φ and all the constraints, be linear then we get a very special sub-class of M.P.P.'s called L.P.P. A mathematical model for an L.P.P. is

Optimise (Max. or Min.) $Z = \sum_{i=1}^{n} c_i x_i$ (1.1.4-a)

subject to
$$\left.\begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \{\leqslant, =, \geqslant\} b_i & (i = 1, 2, ---, m) \\ x_j \geqslant 0 & (j = 1, 2, ---, n) \end{array}\right\} \quad (1.1.4\text{-}b)$$

where a_{ij}, c_i and b_i are assumed to be known constants.

In vector notation, the equivalent L.P.P. is

Optimise (Max. or Min.) $\{CX \,;\, X \in P\}$ (1.1.4-a)'

where

(i) $P = \{X \,;\, AX \{\leqslant, =, \geqslant\} b, \; X \geqslant 0\}$ (1.1.4-b)'

(ii) $A = [a_{ij}]$ is a known mxn matrix.

(iii) $C = (c_1, c_2, \text{------}, c_n)'$ and $b = (b_1, b_2, \text{------}, b_m)'$ are known vectors in E^n and E^m respectively.

Let us now define the following concepts.

Given a system of m simultaneous linear equations in n unknowns,

$$AX = b \quad (m < n) \quad (1.1.5\text{-}a)$$

where rank $(A) = m$ = rank $[A, b]$

and $X \geqslant 0$ (1.1.5-b)

(i) BASIC SOLUTION: If any non-singular mxm matrix B is chosen from A, and if all the (n-m) variables not associated with the columns of this matrix B are set equal to zero, then the solution to the resulting system of equations is called a basic solution to (1.1.5-a)

(ii) BASIS MATRIX: The matrix B is called the basis matrix.

(iii) BASIC VARIABLES: Variables in the basic solution are termed as basic variables.

(iv) NON-BASIC VARIABLES: Variables which are not basic are known as non-basic.

(v) BASIC FEASIBLE SOLUTION (B.F.S.): A basic solution to (1.1.5-a) if it satisfies (1.1,5-b) also is known as a B.F.S.

(vi) DEGENERATE SOLUTION: A basic solution to (1.1.5-a) is degenerate if one or more of the basic variables vanish.

(vii) NON-DEGENERATE SOLUTION: If in a basic solution to (1.1.5-a) none of the basic variables vanishes, then the basic solution is called non-degenerate.

(viii) BAND SOLUTION: A B.F.S. if nondegenerate is called a band solution.

The research work in M.P. can be distinguished with respect to the following three aspects.

(i) THEORETICAL ASPECT: This aspect mainly deals with the development of necessary theory regarding the existence and uniqueness of the optimal solution to a M.P.P. This aspect, though it does not develop a computational technique for obtaining the optimum but is directly responsible for laying down the foundations to develop such procedures. In the present thesis we shall consider this aspect for certain M.P.P.'s.

(ii) COMPUTATIONAL ASPECT: Here we deal with the development of the computational procedures and codes adaptable to high speed computers to obtain the global optimum of a M.P.P. We shall develop, in the present thesis, computational techniques to solve certain class of M.P.P.'s.

(iii) APPLICATIONS: Here our main concern is to reduce certain class of problems in the fields of Economics, Sciences, Technology and other similar fields of interest to M.P.P.'s, to develop necessary theoretical models for those M.P.P.'s and then to solve them if possible by known techniques or to develop new theoretical and computational procedures for them.

Non-linear programming problems can be further divided into following three categories.

1. DETERMINISTIC MODELS: This type of models can be sub-divided into two classes.

(a) DETERMINISTIC CONTINUOUS MODELS: In this class of models, the set of feasible solutions (constraint set) is connected and the O.F. is continuous. The work in the present thesis mainly belongs to this class.

(b) DETERMINISTIC DISCONTINUOUS MODELS: Here the constraint set is not connected and (or) the O.F. is not continuous. The problem of integer programming falls in this class.

2. STOCHASTIC MODELS: In such models the coefficients in the constraints and (or) in the O.F. are random variables.

3. PARAMETRIC MODELS (DYNAMIC MODELS): Here the coefficients in the constraints or (and) in the objective function are dependent on a parameter (e.g. the time) and our interest is to solve the problem for each different value of the parameter.

Another classification depending on the nature of objective and constraint functions will be discussed later.

SECTION - II

BRIEF HISTORY AND REVIEW: Problems of M.P. have been of great interest since long but it is during last two decades only that most of the research work in this direction has been done. With the mathematical formulation of the M.P.P.'s it was soon learnt that the usual classical methods of optimization (e.g. differential calculus and calculus of variations) will not be of great help in solving even a L.P.P., the simplest model of M.P.P.'s. The difficulty stems from the fact that the optimal solutions lie on the boundary of the region of solutions and in L.P.P.'s it is yet worse since it lies at an extreme point. Furthermore, it may be recalled that the methods of differential calculus determine only relative optima which are of considerably less value when global optima are to be obtained. The idea of using the differential calculus to solve an L.P.P. had, therefore, to be discarded and to develop a special procedure the special features of an L.P.P were exploited. It was in 1947, when George B. Dantzig [56], developed the best known and most widely used algebraic iterative procedure, known as 'Simplex Procedure' for solving the L.P.P. The method which was not available till 1951, either gives an exact solution in a finite number of steps or gives an indication for the existence of an unbounded solution. The development of the powerful simplex method, of high speed digital computers, gave a large impetus to the rapid increase of interest to the research workers to give their attention towards the field of M.P.

Research in N.L.P. started almost simultaneously with that in L.P., but encountered considerably greater difficulties. Recently, however, many significant advances of general nature have been made in the area of N.L.P. In 1951, H.W. Kuhn and A.W. Tucker [142], published an important paper entitled, "Non-linear Programming," dealing with the necessary and sufficient conditions for optimal solutions to programming problems, which not only laid down the foundations for a great deal of later work in N.L.P. but also bears a great responsibility for making developments in duality for L.P.P.'s and N.L.P.P.'s. Latter on some generalizations of the work done by Kuhn and Tucker were considered by Arrow, Hurwicz and Uzawa [6,7], Arrow and Enthoven [8], Dorn [67], Kanti Swarup [123, 131], Bector [17,26,31], Mangasarian [148] and Abadie [11].

Consider the program:

$$\text{Maximise} \quad \{\varphi(X) \; ; \; X \in P\} \tag{1.2.1}$$

where the set P is given by

$$P = \{X \; ; \; g_i(X) \leq 0, (i = 1, 2, ----, m)\} \tag{1.2.2}$$

Kuhn and Tucker 142, assuming that all the functions, φ and $g_i, \in C^1$, introduced the following linear systems in m variables $\lambda_1, \lambda_2, -----, \lambda_m$;

$$\lambda_i \geq 0 \qquad i = 1, 2, -----, m \tag{1.2.3}$$

$$\nabla_x \varphi(X_0) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X_0) \tag{1.2.4}$$

$$\sum_{i=1}^{m} \lambda_i g_i(X_0) = 0 \tag{1.2.5}$$

$$g_i(X_0) \leq 0 \qquad i = 1, 2, ----, m \tag{1.2.6}$$

They pointed out that if X_o is an optimal solution to the programming problem (1.2.1), then (1.2.3) through (1.2.5) may have no solution; however, if an additional hypothesis of a very general nature is made, then the compatibility of the system (1.2.3), (1.2.4), (1.2.5) becomes a necessary condition in order that X_o be an optimal solution to (1.2.1). The original Kuhn-Tucker additional hypothesis is known as 'CONSTRAINT QUALIFICATION' property. According to Kuhn-Tucker the Constraint Qualification concept is as follows:

CONSTRAINT QUALIFICATION: Assuming that the functions g_i, $i=1,2,\text{---},m$; $\in C^1$, they are said to satisfy the constraint qualification property if, for any X_o belonging to the boundary of the constraint set P (1.2.2), any vector Y satisfying the homogeneous linear inequalities

$$\left[\nabla_x g_i(X_o)\right]' Y \leq 0 \qquad \text{for all } i \in N_a^o = \left\{ i \,;\, g_i(X_o) = 0 \right\} \tag{1.2.7}$$

is tangent to an arc contained in P i.e. to any vector Y satisfying (1.2.7), there corresponds a differentiable arc $\psi(\theta)$, $0 \leq \theta \leq 1$, contained in P, with $\psi(0) = X_o$ and some positive scalar λ such

that, $$\left[\frac{d\,\psi(\theta)}{d\theta}\right]_{\theta=0} = \lambda Y. \tag{1.2.8}$$

Abadie [11] weakend the above constraint qualification as follows (we shall call them Modified Constraint Qualification).

MODIFIED CONSTRAINT QUALIFICATION: Assume that $X_o \in P$, and that all the functions $g_i \in C^1$ for $i=1,2,\text{---},m$ at point X_o. We say that X_o is 'Qualified' for the system $g_i(X) \leq 0$, $i=1,2,\text{---},m$ in (1.2.2) if, given any vector Y satisfying (1.2.7), there exists an arc $\psi(\theta)$, $0 \leq \theta \leq 1$, contained in P, with $\psi(0) = X_o$, differentiable at $\theta = 0$ and such that $\left[\frac{d\,\psi(\theta)}{d\theta}\right]_{\theta=0} = \lambda Y$ for some positive scalar λ.

Remark: The only difference between the 'Kuhn-Tucker constraint qualification' (1.2.8), and the modified constraint qualification' (1.2.9) is that in the latter we do not require differentiability of any of the function g_i, and of the arc, at any point other than $X = X_0$.

To further weaken the modified constraint qualification (1.2.9), Abadie [11], employed the following theorems and concepts.

TRANSPOSITION THEOREM: Of the following two systems

$$Ax = b$$
$$x \geq 0$$

and

$$u'b > 0$$
$$u'A \leq 0$$

one and only one has a solution, where A is an mxn matrix mx1 column vectors.

TANGENT VECTOR TO A SET: Let V be any given non-empty subset in R^n, $X_0 \in V$, $Y \in R^n$. Then Y is a vector tangent to V at X_0 if there exists a sequence (X_p) contained in V and converging to X_0, and a sequence (λ_p) of non-negative numbers such that the sequence $((X_p - X_0)\lambda_p)$ converges to Y.

CONE OF THE TANGENTS TO A SET: Let V be any given non-empty subset in R^n, $X_0 \in V$ and $Y \in R^n$. If Y is a tangent vector, so is λY for any positive λ (replace λ by $\lambda\lambda_p$ in the above definition), and $Y = 0$, is a tangent vector (consider $X_p = X_0$ and $\lambda_p = 0$ for any p in the definition). The set of all vectors tangent to V at X_0 is then a non-empty cone, which we call the 'Cone of the tangents to V at X_0.

Lemma 1: Let V be any non-empty set in R^n, and X_0 be any point belonging to V; then the cone of the tangents to V at X_0 is a non-empty closed cone [11] .

Let us assume now that $V \equiv P$, (1.2.2). Assume that $X_o \in P$, and that all the functions g_i, $i=1,2,\text{———},m$; $\in C^1$ at X_o; another cone naturally arises namely the closed convex polyhedral cone:

$$\left[\nabla_x g_i(X_o)\right]' Y \leqslant 0 \qquad i \in N_a^o \tag{1.2.7}$$

LINEARIZING CONE: We call the closed convex polyhedral cone;

$$\left[\nabla_x g_i(X_o)\right]' Y \leqslant 0 \qquad i \in N_a^o \tag{1.2.7}$$

the linearizing cone to the system $g_i(X) \leqslant 0$, $i=1,2,\text{———},m$ at X_o.

Lemma 2: Given the set

$$P = \left\{ X ; \; g_i(X) \leqslant 0 , \quad i = 1, 2, \text{---}, m \right\} \tag{1.2.2}$$

and let $X_o \in P$. Assume that all the functions g_i, $i=1,2,\text{———},m$; $\in C^1$ at X_o. Then the linearising cone to P at X_o contains the cone of tangents to P at X_o, [11] .

We now introduce the important concept of Sequential Qualification as given by Abadie [11] , which is weaker than the concept of Kuhn-Tucker constraint qualification.

SEQUENTIAL QUALIFICATION: Given a system

$$g_i(X) \leqslant 0 \qquad i = 1, 2, \text{---}, m \tag{1.2.10}$$

whose set of solutions is P, (1.2.2). Let $X_o \in P$ and assume that all functions g_i, $i=1,2,\text{———},m$; $\in C^1$ at X_o. Then X_o is called 'Sequentially Qualified' for the system (1.2.10) if the cone of tangents to P at X_o is identical with linearising cone to (1.2.10) at X_o.

We now state the following modified form of the most important Kuhn-Tucker theorem, given by Abadie [11] .

Theorem: Let $X_o \in P$ (1.2.2) be sequentially qualified for the system (1.2.10). If all the functions φ , g_i, $i=1,2,\text{———},m$; $\in C^1$ at X_o,

then the necessary conditions that X_o be an optimal (local) solution to program (1.2.1) are that there exist m scalars λ_i^o, i=1,2,-----,m such that

$$\nabla_x \varphi(X_o) = \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(X_o) \tag{1.2.11}$$

$$\sum_{i=1}^{m} \lambda_i^o g_i(X_o) = 0 \tag{1.2.12}$$

$$\lambda_i^o \geq 0 \qquad i=1,2,---,m \tag{1.2.13}$$

Kuhn and Tucker [142] under constraint qualification (1.2.8) showed that if the function φ be a CV function and all the functions g_i be CX functions then for $X_o \in P$, to be an optimal (global) solution to program (1.2.1), the conditions (1.2.11) through (1.2.13) are sufficient also. Kanti Swarup [123,131] has extended the result to the cases in which the objective functions (neither convex nor concave) are respectively linear fractional and indefinite quadratic respectively but the constraints are convex functions. Bector [17,26,31] has considered the cases in each of which the objective function φ, neither convex nor concave (but is more general than a CX or a CV function), is respectively, the product of two non-linear functions, the ratio of two non-linear functions and the ratio of the product of two non-linear functions to the square of a third function, each of the functions being appropriately restricted and all the constraint functions g_i, i=1,2,------,m quasi-convex, and in each of [17], [26], [31], it is shown that for $X_o \in P$, to be a global optimal, conditions (1.2.11) through (1.2.13) are necessary and sufficient. Arrow, Hurwicz and Uzawa [7], Arrow and Enthoven [8], under appropriate

conditions proved that the conditions (1.2.11) through (1.2.13) are sufficient when the function φ is QV (with certain additional restrictions on φ) and all the constraint functions g_i in (1.2.2) are QX. Mangasarian [148] has shown that for $X_0 \in P$ to be the global maximum of program (1.2.1), the conditions (1.2.11) through (1.2.13) are sufficient when the function φ is PCV and all the constraint functions g_i are QX. Extensions of the work done by Kuhn and Tucker [142], to spaces of higher dimensions than R^n are available in [6].

A class of N.L.P. which has been widely studied is one in which the O.F. is non-linear but the constraints are linear. Mathematically the problem may be stated as,

$$\text{Optimize (Max. or Min.)} \quad \varphi(X) \tag{1.2.14}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{1.2.15}$$

where A is an m×n matrix and b is an m×1 column vector.

After the 'Simplex Method' developed for solving a L.P.P. was available in 1951, a special type of N.L.P.P.'s, called, 'Quadratic Programming' attracted a great deal of attention. Mathematical model for a quadratic programming problem is as follows:

$$\text{Minimize} \quad Z = C'X + X'HX \tag{1.2.16a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{1.2.16b}$$

Where H is an nxn positive semi-definite matrix, and matrix A and vector b are as in (1.2.15).

In 1955 and afterwards a number of papers appeared on (1.2.16) They are, to include the works of Barankin and Dorfman, [12,13], Beale [14], Dennis [60], Frank and Wolfe [73], Hildreth [110], Houthakker [111], Lemke [145], Markowitz [153], Van de Panne and Theil [174], Van de Panne and Whinston [175, 176], Boot [37,39,40], Wolfe [180], and Frisch [75,76].

Another special class of N.L.P. is that of 'Convex Programming' which has been paid attention. It is stated as follows:

Optimize $\varphi(x)$ (1.2.17-a)

subject to

$$g_i(x) \leq 0 \qquad i = 1,2,\dots,m \tag{1.2.18-b}$$

where the function φ is CX and all g_i, i=1,2,——,m are also convex functions.

If we assume

$$P = \left\{ x \; ; g_i(x) \leq 0 \, , i = 1,2,\dots,m \right\} \tag{1.2.2}$$

then the following well known results, which can, for example, be found in Hadley [105], hold for the problem (1.2.18).

1. Every local minimum of φ over P is a global minimum (1.2.19)
2. The set of points in P, at which φ takes on its global minimum is a convex set. (1.2.20)
3. If the global maximum of φ over P is taken on in the interior of the set P, then the function φ is constant over P. (1.2.21)

4. If the set P be compact also, then the global maximum of φ over P will be taken on at one or more extreme points of P. (1.2.22)

If all the functions φ, g_i, $i=1,2,\text{------},m$; $\in C^1$ and the Kuhn-Tucker constraint qualification (1.2.8) are satisfied then conditions

$$\nabla_x \varphi(x_o) + \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(x_o) = 0$$

$$\sum_{i=1}^{m} \lambda_i^o g_i(x_o) = 0$$

$$g_i(x_o) \leq 0 \qquad i = 1, 2, ---, m$$

$$\lambda_i^o \geq 0 \qquad i = 1, 2, ---, m$$

for some λ_i^o, $i=1,2,\text{------},m$; are necessary and sufficient for $X_o \in P$ to be global minimum of (1.2.18). (1.2.23)

Several non-finite algorithms exist for the solution of a convex programming problem. The most prominent of those are,

1. Zoutendijk's method of feasible directions [185].
2. Rosen's gradient projection method [155,156].
3. Frich's multiplex method [9].
4. Kelley's cutting plane method [136].

Many scientists had endeavoured to extend the power of simplex and similar methods to other mathematical programming problems, and considerable results have now been achieved. In addition to L.P. and quadratic programming in N.L.P., there are two more particular types of N.L.P. named linear integer programming and the programming with separable convex functionals to which the power of adjacent vertex

vertex method has been extended. In linear integer programming, we have for an L.P.P., an additional requirement that the variables can taken only integral values. R.E. Gomory is well known for doing a good deal of work in this direction. In 1954, A. Charnes and C.E. Lemke [49] published a paper in which they had approximated a convex programming problem by L.P.P. to which simplex method could be successfully applied for obtaining an approximate solution.

The kind of advancement naturally raises the question how far this power extends. This task has been set explicitly set by Charnes and Cooper in 1956, [46], who referred also to the possibility that the extreme point methods e.g. the simplex method, may be capable of solving problems with objective functions which are neither convex nor concave. Bela Martos [151, 152*] furnished a partial answer to this question for the type of mathematical programming problems given by,

$$\text{minimise} \quad \{ \varphi(X) \; ; \; X \in L \} \tag{1.2.24}$$

where it is assumed that,

(i) $$L = \{ X \; ; \; AX \leq b, \; X \geq 0, \; X \in R^n \} \tag{1.2.25}$$

A and b being as in (1.2.15).

(ii) L contains two different points.

(iii) L is bounded.

(iv) The objective function φ (as already assumed in the begining) is continuous in L.

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* The author received the reprint of this paper from Prof. Bela Martos in July, 1968, when the author had already completed his work included in the thesis.

Bela Martos [151,152] introduced the concept of EQX (EQV) functions which lie between the class of SQX (SQV) functions and class of QX (QV) functions. Bela Martos [151] also introduced the idea of QM(SQM) functions, and proved the following most important results.

1. The necessary and sufficient condition for attainment of a global minimum of φ on an extreme point of L are that φ is QV in L and EQX between any two points of L which do not lie on the same edge of L*.

2. The family of functions φ which have all derivatives and which can be optimised by adjacent vertex methods in (1.2.24) consist of the quasi-monotonic functions. This is all such functions and only such functions that can be treated in this manner for purposes of global optimisation.

Mangasarian [148] introduced the concept of Pseudo-Convex (Pseudo-Concave) functions and established the following results for PCX functions and stated that analogous results hold for PCV functions.

1. Let φ be PCX on a set $B \subset R^n$. If, $\nabla_x \varphi(X_0) = 0$, then X_0 is a global minimum over B. (1.2.26)

2. Let φ be CX on a convex set $S \subset R^n$. Then φ is PCX on S, and not conversely. (1.2.27)

3. Let S be a convex subset in R^n. If φ is PCX on S, then φ is SQX (and hence QX) on S, but not conversely. (1.2.28)

4. Let S be a convex set in R^n. If φ be PCX on S, then every local minimum of φ over S is global minimum also. (1.2.29)

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* Correct version of result in [151] given in [152].

Remark: It may be remarked over here that to prove (1.2.29), Mangasarian [148] has made use of (1.2.20), but in the present thesis (1.2.29) has been proved without using (1.2.20).

6. Let in program given by (1.2.1) and (1.2.2), φ be a CV function and all g_i be QX functions and let all φ and g_i, $i=1,2,\text{———},m$; $\in C^1$. If there exist an $X_0 \in P$ and λ_i^0, $i=1,2,\text{———},m$ satisfying the Kuhn-Tucker differential conditions (1.2.3) through (1.2.6) then

$$\varphi(X_0) = \max\left\{ \varphi(X) \; ; \; g_i(X) \leq 0, \; i=1,2,\cdots,m \right\} \qquad (1.2.30)$$

Recently in 1967, Ponstein[1] [148a] established the relations between different kinds of convexity of functions and presented a counter example which ensured that a local minimum (maximum) of a QX(CV) function may not be a global minimum (maximum). Furthermore, he proved that if S be a convex set in E^n and if φ be a SQX function in S, then every local minimum of φ over S is global minimum also. Bector extended the results (1.2.19) through (1.2.23), of convex programming, to more general problems of, non-linear indefinite functional programming [17], non-linear fractional functional programming [26], non-linear indefinite-fractional functional programming [31], and the results (1.2.19) through (1.2.22) for explicit quasi-convex programming [32] (a problem in which we optimize an SQX function on a convex set). Furthermore, he introduced the concept of some new functions of which Strong Pseudo-Concave (convex) Functions [33], are of much interest. Some of the properties of such functions have also been established in [33].

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1. This paper 148a was brought to the notice of the author when he had already finished the work contained in 32 .

SPECIAL CASES OF INTEREST:

Most of the following work is concerned with the study of theoretical developments, leading to the development of computational techniques, somewhat similar to already existing method, for special type of N.L.F.P.'s called, (i) Non-linear Indefinite Functional Programming Problems (N.L.I.F.P.P.'s),

(ii) Non-Linear Fractional Functional Programming Problems (N.L.F.F.P.P.'s),

(iii) Strong Pseudo-Concave Programming Problems (SPV P.P.'s),

(iv) Pseudo-Concave Programming Problems (PV P.P.'s). Furthermore, some theoretical results of fundamental importance have been established for Explicit Quasi-Concave Programming (EV P.P.). A few properties of newly introduced strong pseudo-concave (convex) functions are investigated.

INDEFINITE FUNCTIONAL PROGRAMMING:

A N.L.I.F.P.P. of the type considered in the present work is stated as

$$\text{Maximize} \quad \left\{ \varphi(x) = f(x)\, g(x) \; ; \; x \in P \right\} \tag{1.2.31}$$

where P is a convex set in R^n, and, if necessary, constrained by specified constraint functions, and f and g are strictly positive functions over P and assumed to be concave over a convex set S, $P \subset S \subset R^n$.

It will be appropriate here to review briefly the already existing theoretical results and some of the computational techniques developed to solve a particular type of N.L.I.F.P., called Indefinite Quadratic Programming (I.Q.P.). The problem of I.Q.P., first considered by A. Orden [154], may be stated as follows,

Minimize (or Maximize) $\varphi(X) = C'X + X'HX$ (1.2.32)

subject to $AX = b, \; X \geq 0$ (1.2.33)

Where,

(i) H is an nxn matrix.

(ii) A is mxn matrix.

(iii) C and b are respectively nx1 and mx1 column vectors.

A. Orden [154] developed a method which includes only equality constraints $AX = b$, later on several authors [124], [139], [159], [179], succeeded in providing techniques with non-negative restrictions on the decision variables (i.e. $X \geq 0$), but involving great computational efforts in some cases.

Kanti Swarup in his papers [124, 125, 126, 131, 132] considered particular type of I.Q.P. (which can be treated as particular case of the problem considered in the present thesis). Such problems were obtained in a variety of context. General mathematical model considered by Kanti Swarup in [124, 125, 126] is given as:

Maximize $\varphi(X) = (C'X + \alpha)(d'X + \beta)$ (1.2.34-a)

subject to

$$AX \leq b$$
$$X \geq 0$$ (1.2.34-b)

Where A, C, b are as already explained, d is an nx1 column vector, α, β are arbitrary scalar constants, and $C'X + \alpha$, $d'X + \beta$ are strictly positive over the set of feasible solutions.

Such problems are encountered in a variety of contexts. One such problem could arise [128] in a situation of competitive market

in which there are two competitors for a given product whose total demand is constant, the sale of product depends linearly on the market prices and our interest is to maximize the profit of one of the competitors. If, however, the sale of the product depends non-linearly on the market prices, the problem becomes that of N.L.F.P.

In [124], problem (1.2.34) for its solution has been replaced by a convex programming problem, with the help of homeomorphisms transformation [45], from which follows the globality of local optima with indefinite quadratic function. Using transformation $Y = tX$ where $t > 0$ [45], to solve the problem (1.2.34), we can solve

$$\text{Minimise} \quad \psi(t,Y) = \frac{t^2}{C'Y + \alpha t}$$

subject to

$$AY - bt \leq 0$$
$$d'Y + \beta t = 1$$
$$t, Y \geq 0$$

and Max $\varphi(X) = \varphi(Y^*/t^*)$, where $\psi(t^*, Y^*) = \min \psi(t,Y)$

In [125], a parametric linear programming approach, similar to that given by Joksch [115], has been provided for the solution of problem (1.2.34). Conditions for extrema are derived and the possibilities for local and global extrema are discussed.

Let $C'X + \alpha$ be a linear function to be maximized (or minimized) under the given constraints and $d'X + \beta = \lambda$.

The optimal solution as a function of λ is,

$$Z^+(\lambda) = \max\,(C'X + \alpha)$$

subject to

$$A X \leq b$$
$$X \geq 0$$
$$d' X + \beta = \lambda$$

Also $$Z^-(\lambda) = \min (c' X + \alpha)$$

subject to

$$A X \leq b$$
$$X \geq 0$$
$$d' X + \beta = \lambda$$

If
$$\left.\begin{aligned} Z^+(\lambda) &= u_i^+ + \lambda v_i^+ \\ Z^-(\lambda) &= u_i^- + \lambda v_i^- \end{aligned}\right\} \quad \lambda_i \leq \lambda \leq \lambda_{i+1}$$

For the maximization, we consider

$$\text{Max. } Z^+(\lambda)\lambda \qquad \lambda > 0$$

and for minimization,

$$\text{Min. } Z^-(\lambda)\lambda \qquad \lambda > 0$$

where u_i, v_i are obtained by solving the problem as a parametric linear programming [80].

In another paper [126], Kanti Swarup developed a finite iteration procedure for finding local minimum (which may not be global minimum) of problem (1.2.34). The problem was attacked directly beginning with a B.F.S. and showing the conditions under which the solutions can be improved. The method followed is exactly similar to 'Simplex Technique' in L.P. In Theorem 1 of the paper [126], in (1.2.34) was established to be EQV in feasible area, which paved the way for the existence of simplex-like technique for its solution. The optimality conditions for local optimum were derived.

In [131] , Kanti Swarup considered the Indefinite Quadratic Program,

Maximise $\varphi(X) = (C'X+\alpha)(d'X+\beta)$ (1.2.35-a)

subject to

$$g_i(X) \leq 0 \qquad (i = 1,2,\text{----},m) \tag{1.2.35-b}$$

where $g_i(X)$ are CX functions and $C'X+\alpha$, $d'X+\beta$ are strictly positive over the set of feasible solutions. Sufficient conditions, in terms of Lagrange multipliers for global maximum of (1.2.35) are derived. In another paper [128] , Kanti Swarup introduced 'Indefinite Quadratic Programming' as given in (1.2.35) with all the g_i, $i=1,2,\text{------},m$; $\in C^1$, and CX, as a class of Pseudo-Concave Programming. This gives rise to an important result, "Kuhn-Tucker differential conditions are sufficient for optimality, when the objective function is indefinite quadratic and constraints are quasi-convex". Further, a result in the direction of duality, viz. the 'Converse Duality Theorem' was also provided.

Kanti Swarup [132] , provides a procedure, "in two phases" for maximising an indefinite quadratic function subject to one quadratic constraint and a number of Linear constraints. The method combines the technique of solving the indefinite quadratic programming with linear constraints and that of maximizing the convex quadratic function over a convex region. Bector [34] , under appropriate assumptions, developed a computational technique similar to, "Method of Feasible Directions" [184,185] , to obtain the global maximum of N.L.I.F.P.P. (1.2.31) with the constraint set P given by,

$$P = \{X \, ; \; g_i(X) \leq 0 \; , \; i = 1,2,\text{---},m, X \in S\}$$

where g_i, $i=1,2,\text{------},m$ are QX functions each $\in C^1$.

FRACTIONAL FUNCTIONAL PROGRAMMING:

The other class of programming problems considered in the present work is N.L.F.F.P. of which L.F.F.P. follows as a particular case. The general mathematical model for N.L.F.F.P. is

$$\text{Optimize (Max. or Min.)} \left\{ \varphi(x) = \frac{f(x)}{g(x)} \; ; \; x \in P \right\} \tag{1.2.36}$$

where P is a convex set in R^n, and if necessary constrained by specified constraint functions, and f and g are respectively CV (or CX) and CX (or CV), and $f \geqslant 0$, $g > 0$ over constraint set P, (if g be linear then non-negativity restriction on f is not necessary). When both f and g are linear then only g being strictly positive over P, and if the set P also be a convex polyhedral set, then the problem (1.2.36) is called Linear Fractional Functional Programming (L.F.F.P.)

Fractional Functional Programming Problems (both linear and non-linear) arise in a variety of context. One such problem could occur [121] , in an industry , involving situations of diminishing returns, constant returns, and increasing returns which are determined by elasticity of total, or the ratio of marginal cost to average cost. Another problem could arise in Inventory Control and Production also.

L.F.F.P. is considered to be the simplest possible case of fractional functional programming which admists of a finite iteration technique for its solution. Such problems arise, in the distribution of fire over enemy targets, trim problem [114, 81] where the objective

is to minimize the average waste, and in the optimal maitenance and repair policies in the context of a Markoff process formulation [137].

Mathematical model for L.F.F.P. is

Maximise $$Z = \frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta} \qquad (1.2.37\text{-a})$$

subject to
$$\left.\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j &\leqslant b_i \quad (i = 1, 2, ---, m) \\ x_j &\geqslant 0 \quad (j = 1, 2, ---, n) \end{aligned}\right\} \qquad (1.2.37\text{-b})$$

In vector notation it may be written as

Maximise $$Z = \frac{c'X + \alpha}{d'X + \beta} \qquad (1.2.37\text{-a})$$

subject to

$$\left.\begin{aligned} AX &\leqslant 0 \\ X &\geqslant 0 \end{aligned}\right\} \qquad (1.2.37\text{-b})$$

where A,b,c,d, α and β are as in problem (1.2.34).

In 1956 Isbel and Marlow [114] established a convergent iterative process which involved replacing the ratio by the problem of optimising a sequence of different linear functionals provided the denominator in (1.2.37-a) does not vanish.

Bela Martos [149,150] in 1960 in his paper, "Hyperbolic Programming" considered the problem (1.2.37). The motivation for calling the problem as Hyperbolic Programming [150], comes from the fact that the graph of the function $y = \frac{cx + \alpha}{dx + \beta}$ in two dimensional plane is a hyperbola. Bela Martos solved the problem (1.2.37) with the help of modified simplex technique. The paper is mainly discussed in two cases. (1) Simple Case and (2) General Case.

SIMPLE CASE: The hyperbolic programming problem is of simple case if it satisfies the following two conditions.

(i) The set of feasible solutions, to be denoted by P, is a bounded set, and, therefore, is a compact convex polyhedral set.

(ii) The denominator of the O.F. is not zero on any point of the constraint set, P.

Firstly it has been proved that the maximum of the problem will occur at the vertex of P. He then attacks the problem by using simplex procedure starting from a B.F.S. and derives that the local maximum so obtained is global maximum also. Sufficient conditions for optimality to L.F.P.P. are established.

GENERAL CASE: In general case when the conditions for the simple case are absent the problem may have an optimal solution even though the set P may be unbounded or the denominator becomes zero over some points of P. Here, he considers,

(i) the 'good point' of the set P, where

$$\left.\begin{array}{ll} \text{either} & d'X+\beta > 0 \\ \text{or} & d'X+\beta = 0 \end{array}\right\} \quad \text{and} \quad c'X+\alpha < 0$$

(ii) the 'bad point' of the set P, where

$$\left.\begin{array}{ll} \text{either} & d'X+\beta < 0 \\ \text{or} & d'X+\beta = 0 \end{array}\right\} \quad \text{and} \quad c'X+\alpha > 0$$

(iii) the 'singular point' of the set P, where

$$d'X+\beta = 0 \ , \quad c'X+\alpha = 0$$

It has also been remarked that the degeneracy problem arising in the course of hyperbolic programming problem can be solved with the aid of usual method described by Charnes [47].

In his paper [151] in 1965, "The Direct power of adjacent vertex programming methods," Bela Martos proved the L.F.F. to be QM, as a result of which it follows that for a L.F.F.P., when the constraint set is a non-empty compact convex polyhedral set, the maximum will occur at a vertex of the constraint set and every local maximum is a global maximum.

Charnes and Cooper [45], in 1962 in their paper, "Programming with linear fractional functionals" have replaced a L.F.F.P.P. with at most two ordinary L.P.P.'s under the assumption that the set of feasible solutions is regular. With the help of the transformations $Y = tX$, which are homeomorphisms they have shown that problem (1.2.37) can be solved by solving one of the following two L.P.P.'s

Maximise $\Psi(t, Y) = C'Y + \alpha t$

subject to

$$\left.\begin{aligned} AY - bt &\leq 0 \\ d'Y + \beta t &= 1 \\ t, Y &\geq 0 \end{aligned}\right\} \tag{1.2.38-a}$$

and

Maximise $\Psi(t, Y) = -C'Y - \alpha t$

subject to

$$\left.\begin{aligned} AY - bt &\leq 0 \\ -d'Y - \beta t &= 1 \\ t, Y &\geq 0 \end{aligned}\right\} \tag{1.2.38-b}$$

provided the sign of the denominator at the optimal solution is known.

Dorn [67] has discussed the problem in four different categories as under:

(1) The program has a finite maximum at a finite point.

(2) The program has an infinite maximum at a finite point.

(3) The program has a finite maximum at a non-finite point.

(4) The program has an infinite maximum at a non-finite point.

Furthermore Dorn [67] has shown that Kuhn-Tucker differential conditions (1.2.11) through (1.2.13) are sufficient for the existence of the optimal solution and he has developed a generalization of Lemke's dual algorithm for L.P. to obtain the solution of L.F.P.

Dinkelbach [61] , in 1962, gave the solution of L.F.F.P. by just modifying the simplex method. He also has indicated the possibility of application of this type of problems to the theory of games with non-linear pay-off functions.

Gilmore and Gomory [81] in 1963 introduced the idea of rational objective functions when customers' orders are not for fixed amounts, but rather for a range of amounts. The method of solution of rational O.F. with linear constraints, considered by the authors, is just similar to that of Bela Martos [150] . Kanti Swarup [122] has discussed mainly three aspects. In the first part of the paper he has discussed a computational technique for solving the L.F.F.P. The technique is similar to that of Beale [16] . Second part contains certain relations and common characteristics of L.F.F.P. and its equivalent linear programming problem (1.2.38). Third part of the paper is devoted to finding an algorithm for solving L.F.F.P.P., which is similar to dual simplex algorithm for L.P., furthermore, outlines of the technique

for obtaining an integer solution to L.F.I.P.P., which is exactly similar to that of Gomory [88-92] , are given. In another paper [120] Kanti Swarup has solved L.F.F.P.P. (1.2.37), under the assumption that denominator of the O.F. is strictly positive of the set of feasible solutions assumed to be regular, by adopting exactly the approach given by Hadley [104] to solve a L.P.P. In still another paper [121] , he has generalised the L.F.F.P. to a programming problem with quadratic fractional functionals, in which the objective function is the ratio of two quadratic function, and the constraints are linear. Following Charnes and Cooper [45] exactly, Kanti Swarup has replaced the problem by atmost two programming problems, with quadratic objective functions differing from each other by only a change in sign, the constraints being linear and one quadratic constraint. The quadratic constraints in each of the two problems differ from each other in sign. The difficulty in the problem dealt with by Kanti Swarup [121] , is that from the reduced problems in [121] , the equivalent linear programming problems of [45] , could not be obtained as a particular case.

H.C. Joksh [115] reduced such problems to parametric linear programming by treating the value of one of the linear forms in the O.F. as a parameter. Conditions for the extrema are derived. R. Jagannathan [117] developed some results, concerning the relationship between fractional and parametric programming, leading to computational possibilities for L.F.F.P. The results of Jagannathan have been further investigated by Dinkelbach [62] for non-linear fractional functionals and it is shown as to how the algorithm to solve such a problem can be

related to the method of Isbell and Marlow [114] and the quadratic parametric approach by Ritter [158].

Kanti Swarup [123] considered the problem

$$\text{Maximise} \quad Z = \frac{c'X + \alpha}{d'X + \beta} \tag{1.2.39-a}$$

subject to

$$g_i(X) \leq 0 \quad , \; i = 1, 2, \ldots, m \tag{1.2.39-b}$$

where all the functions g_i are convex, and established necessary and sufficient conditions, similar to Kuhn-Tucker differential conditions (1.2.11) through (1.2.15), for the global maximum. Furthermore, in [130], he has considered some aspects of duality of (1.2.39).

Aggarwal [3], extended the L.F.P.P. to "Standard Error Fractional Functionals Programming". The mathematical formulation of such a problem is

$$\text{Maximise} \quad Z = \frac{c'X - (X'HX)^{1/2} + \alpha}{d'X + \beta} \tag{1.2.40-a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{1.2.40-b}$$

where, H is an $n \times n$ real symmetric positive semi definite matrix and other symbols as usual. By employing the homeomorphic transformations $Y = tX$ and following Charnes and Cooper [45], exactly, Aggarwal has reduced (1.2.40) to

$$\text{Maximise} \quad \psi(t, Y) = c'Y - (Y'HY)^{1/2} + \alpha t$$

subject to

$$\begin{aligned} AY - bt &\leq 0 \\ d'Y + \beta t &= 1 \\ t, Y &\geq 0 \end{aligned}$$

and

Maximize $\quad \psi(t, Y) = -c'Y + (Y'HY)^{1/2} - \alpha t$

subject to

$$\begin{aligned} AY - bt &\leq 0 \\ -d'Y - \beta t &= 1 \\ t, Y &\geqslant 0 \end{aligned}$$

which are of the form discussed by Sinha [163] .

Kanti Swarup and Aggarwal [134] developed a computational technique exactly on the lines of Van de Panne [177] for a L.F.P. subject to one quadratic and a number of linear constraints. The problem is discussed in two phases. Phase I is concerned with the maximization of L.F.F. subjected to linear constraints and Phase II deals with parametric version of quadratic programming.

Aggarwal [2] deals with the problem of Maximizati n (locally) of a special type of convex function with linear constraints. The problem is

Maximize $\quad \varphi(X) = \dfrac{(c'X)^2}{d'X + \beta}$ (1.2.41-a)

subject to

$$\left.\begin{aligned} AX &= b \\ X &\geqslant 0 \end{aligned}\right\} \qquad (1.2.41\text{-b})$$

where it is assumed that the constraint set is regular and $d'X+\beta>0$ over the constraint set. A technique similar to 'Simplex Procedure' is developed to obtain the local maximum of the problem (1.2.41). In another paper [1], Aggarwal considered the minimization of the function, $\dfrac{X'HX + c'X + \alpha}{d'X + \beta}$, subject to linear constraints.

Under the assumptions that the matrix H is nxn real symmetric positive-semi definite matrix and $d'X + \beta > 0$ over the set of feasible solutions, he proved the O.F. to be SQX.

Gogia [98, 97], developed respectively the revised simplex method and multiplex method for solving a L.F.F.P.P. In another paper [99], he developed an approximation technique, similar to that of Charnes and Lemke [49], for obtaining the solution of a general non-linear fractional functional program. Bector [36] developed a computational technique similar to 'Method of Feasible Directions' [184, 185], for obtaining the global optimum of N.L.F.F.P.P. of the type (1.2.36), in which the constraint set P is assumed to be constrained by appropriately restricted QX functions which belong to C^1.

STRONG PSEUDO-CONCAVE PROGRAMMING (PSEUDO-CONCAVE PROGRAMMING):

In the programming problem

Optimise $\{\varphi(X) \; ; \; X \in P\}$

if the function φ be a SPCV (PCV) function, it being assumed that $P \subset S$ is a convex set, then we say that the above problem is that of S.V P.P. (PV P.P.). Idea of SPCV functions is introduced only in the present thesis and the idea of PCV functions was introduced very recently in 1965 by Mangasarian [148], with their applications to fields of Mathematical Programming & Stability Theory Concerning PV P.P. also very few papers are available so far. They are to include the names of Ponstein [148a], Gupta and Bector [96], Kanti Swarup [128], Bector [32,33]. In the present thesis a

a computational technique, similar to 'Method of Feasible Directions' is developed to solve a SPV P.P. (PV P.P.).

EXPLICIT QUASI-CONCAVE PROGRAMMING: A programming problem stated as

$$\text{Optimize} \quad \left\{ \varphi(X) \ ; \ X \in P \right\}$$

is said to be that of EV P.P., if the O.F. φ be EQV, it being assumed that $P \subset S$ is a convex set, if necessary constrained by appropriately restricted functions. Recently, in 1965, Bela Martos [151] introduced the notion of EQV functions the concept of explicit quasi-concavity according to Bela Martos [151] is intermediate between quasi-concavity and strict quasi-concavity. In the present thesis a few properties of EQV and EQX functions, with special reference to mathematical programming, have been studied. No paper on Explicit Quasi-Concave (Convex) Programming seems to have appeared so far except that we can include the papers of Arrow and Enthoven [8], Arrow Hurwicz and Uzawa [7] and Kovacs [141] who discussed certain mathematical programming problems containing QX functions (more general than EQX functions). The first two papers are primarily concerned with the generalizations of the basic Kuhn-Tucker theorems and the third one deals with the extension of Rosen's gradient projection method to quasi-concave maximization. DeFinetti [59], Fenchel [72], Berge [41] and Deak [58] also have investigated significant characters of QX and (or) QM functions without reference to mathematical programming problems. In the present thesis we characterize, the problems of N.L.I.F.P. and N.L.F.F.P. and the type of problems considered by Geofferin [101], when at least one

of the suitably restricted functions involved in the O.P. is non-differentiable on the constraint set, as Explicit Quasi-Convex Programming problems.

Duality in Non-Linear Programming has been since long of interest. An extensive use of duality has been made in theoretical and computational applications. It is well known that the duality principles in convex programming, connect two programming problems, one of which called the primal problem (to be denoted by (PP) in the analysis to follow) is a constrained maximization problem (or minimization) problem and the other called the dual problem (to be denoted by (DP) in the analysis to follow), is a constrained minimization (or maximization) problem in such a way that the existence of a solution to one of these problems ensures a solution to the other and the extrema of the two problems are equal.

Wolfe [181] , gives the following theorem, "If the solution to the (PP)

$$\left.\begin{array}{l}\text{Minimise } \varphi(X) \\ \text{subject to} \\ \qquad g_i(X) \geqslant 0 \qquad i = 1, 2, \cdots, m\end{array}\right\} \qquad (1.2.45)$$

exists, then the solution to the (DP)

$$\left.\begin{array}{l}\text{Maximise } \quad F(X,\lambda) = \varphi(X) - \sum_{i=1}^{m} \lambda_i g_i(X) \\ \text{subject to} \\ \qquad \nabla_x \varphi(X) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X) \\ \qquad \lambda_i \geqslant 0 \qquad i = 1, 2, \cdots, m\end{array}\right\} \qquad (1.2.46)$$

exists and the two extrema are equal, where φ is a CX function in C^1 and all $g_i \in C^1$ and are CV functions.

M.A. Hanson [108] established, under suitable assumptions, a theorem similar to that of Wolfe and its converse for the following pair of problems.

$$\left.\begin{array}{ll} \text{(PP)} & \text{Minimise } \varphi(X) \\ & \text{subject to} \\ & \quad g_i(X) \geqslant 0 \qquad i = 1, 2, \ldots, m \\ & \quad X \geqslant 0 \end{array}\right\} \quad (1.2.47)$$

$$\left.\begin{array}{ll} \text{(DP)} & \text{Maximise } F(X,V) = \varphi(X) - X'\nabla_x \varphi(X) - V'[g(X) - X'\nabla_x g(X)] \\ & \text{subject to} \\ & \quad \nabla_x V' g(X) - \nabla_x \varphi(X) \leqslant 0; \quad V \geqslant 0 \end{array}\right\} \quad (1.2.48)$$

where $g(X)$ is an m-dimensional concave function of X; V is an m-dimensional Lagrange Multiplier and $X'\nabla_x g(X) = (X'\nabla_x) g(X)$ is an m-vector, $X'\nabla_x$ is a scalar operator.

Kanti Swarup [119] , established the following theorem,

" If the solution to the (PP)

$$\left.\begin{array}{l} \text{Minimise } \varphi(X) \\ \text{subject to} \\ \quad g_i(X) \geqslant 0 \qquad i = 1, 2, \ldots, m \\ \quad x_j \geqslant 0 \qquad j = 1, 2, \ldots, n \end{array}\right\} \quad (1.2.49)$$

exists then the solution to the (DP)

$$\left.\begin{aligned}
&\text{Maximise } F(x,\lambda) = \varphi(x) - \sum_{i=1}^{m} \lambda_i g_i(x) \\
&\text{subject to} \\
&\qquad \frac{\partial \varphi}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \geq 0 \qquad j = 1, 2, \cdots, n \\
&\qquad x_j \left[\frac{\partial \varphi}{\partial x_j} - \sum_{i=1}^{m} \lambda_i \frac{\partial g_i}{\partial x_j} \right] = 0 \qquad j = 1, 2, \cdots, n \\
&\qquad \lambda_i \geq 0 \qquad i = 1, 2, \cdots, m
\end{aligned}\right\} \quad (1.2.50)$$

exists and the two extrema are equal, where φ and all g_i are as in Wolfe [181] .

Mangasarian [147] , and Huard [109] , proved the converse of duality theorem established by Wolfe [181] . Kanti Swarup [130] extended the theorem of Mangasarian [147] and the converse duality theorem of Huard to the case in which the objective function is L.F.F. and the constraint functions are convex functions. In the present work, the converse duality theorem is proved for N.L.I.F.P. and N.L.F.F.P. problems, which are more general problems than considered by works of [147,109,130] .

SECTION - III

SUMMARY OF THE THESIS: We now give below a brief description of the contents of the various chapters of this thesis as the relevant literature.

Whenever we start tackling a problem, there are always certain fundamental questions, for example, the existence of the solution, the uniqueness of the solution etc. which occupy our mind. In M.P. also we are to reply certain questions of fundamental nature before we

venture forth to formulate an optimism to develop a computational technique to solve some M.F.P. Those questions are, generally, of the type:

(i) Whether the optimal solution to the problem exists.

(ii) If the optimal solution exists, shall we obtain the global optimal with the help of the technique developed.

(iii) The global optimum, if it exists, is it unique.

(iv) Duality aspects of the problem.

Chapter II answers such fundamental questions for the problem of N.L.I.F.P., whereas Chapter III deals with the problem of Fractional Functional Programming, which are more general than convex programming problems.

In Chapter II, the following N.L.I.F.P.P. is considered.

Maximize (or Minimize) $\varphi(x) = f(x)\, g(x)$ for $x \in P$. (1.3.1)

where,

(i) The set $P \subset S$ is a non-empty, closed convex set, and if needed, assumed to be constrained by a system of functions, given by

$$g_i(x) \leq 0 \qquad i = 1, 2, \text{----}, m \tag{1.3.2}$$

each g_i, $i=1,2,\text{------},m$; defined to be QX over S such that the set

$$P = \left\{ x \,;\ g_i(x) \leq 0 \ ,\ i = 1, 2, \text{---}, m \,;\ x \in S \right\} \tag{1.3.3}$$

is a convex set.

(ii) f and g are CV functions on a convex set S in E^n, and are strictly positive over the set of feasible solutions P, such that the objective function φ is neither convex nor concave.

(iii) The program has a finite optimum at a finite point of P.

The chapter is divided into four sections. Section I deals mainly with the extension of the results (1.2.19) through (1.2.22) for N.L.I.F.P.P. (1.3.1) i.e.

1. Every local maximum of φ over P is global maximum.

2. The set of points in P, at which φ takes on its global maximum is a convex set.

3. If the O.F. φ takes on its global minimum in the interior of the set P, then the objective function φ is constant throughout P.

4. If the global minimum of φ over P, assumed to be bounded, be finite, then it is taken on at one or more of the extreme points of the set P.

In Section II, the Kuhn-Tucker existence theorem is proved for program (1.3.1) where O.F. is neither CX nor CV and when P is given by (1.3.3), Viz. "If $X_o \in P$, be sequentially qualified for the system (1.3.2) and if all the functions f, g, g_i, $i=1,2,------,m; \in C^1$ at X_o, then the necessary and sufficient conditions that X_o be an optimal solution to the program (1.3.1) are that there exist m scalars λ_i^o, $i=1,2,------m$, such that

$$\nabla_x \varphi(x_o) = \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(x_o)$$

$$\sum_{i=1}^{m} \lambda_i^o \, g_i(x_o) = 0$$

$$g_i(x_o) \leqslant 0$$

$$\lambda_i^o \geqslant 0$$

Furthermore another theorem which states that if any point $X_0 \in P$, possess any one of the properties of being a stationary point, a local maximum, and, a global maximum, it possess the other two also, is proved.

Section III is devoted to the discussion of certain aspects of duality of problem (1.3.1). The main result proved is the converse duality theorem similar to that of Mangasarian [147], Huard [109].

Lastly in Section IV, a few particular cases considered by the author and other research workers have been discussed.

Chapter III is devoted to the following fractional functional programming problem:

$$\text{Optimise (Maximize or Minimize) } \varphi(X) = \frac{f(X)}{g(X)} \quad \text{for } X \in P. \tag{1.3.4}$$

The chapter is divided into five sections. Sections I through III contain the N.L.F.F.P.P.

$$\text{Optimise } \varphi(X) = \frac{f(X)}{g(X)} \qquad \text{for } X \in P \tag{1.3.5a}$$

Where,

(i) $P \subset S$ is a non-empty closed convex set and if necessary assumed to be constrained by a system of functions given by

$$g_i(X) \leq 0 \quad i \in M = (1, 2, \ldots, m) \tag{1.3.5b}$$

all g_i, $i \in M$ are QX, (and if needed, assumed to be $\in C^1$), such that the set

$$P = \left\{ X \; ; \; g_i(X) \leq 0, \; i \in M \; ; \; X \in S \right\} \tag{1.3.5c}$$

is a convex set.

(ii) f is non-negative and convex over S, g is strictly positive and concave over S, (If g be linear, then condition of non-negativity on f can be omitted).

(iii) The program has a finite optimum at a finite point of P.

In Sections I and II the following theoretical results of fundamental importance are established.

1. Every local minimum of φ over P is a global minimum also.

2. The set of those points in P, at which φ takes on its global minima, is a convex set.

3. If the O.F. φ has a point of global maximum in the interior of the set P, then the O.F. φ is constant over P.

4. The function φ takes on its global maximum at one or more of the extreme points of the set P.

5. If $X_0 \in P$ (1.3.5c) be a point which is sequentially qualified, [11], for the system (1.3.5b) and if all the functions f,g and g_i, $i \in M$, $\in C^1$ at X_0, then the necessary and sufficient conditions for the existence of a minimum to the program given by (1.3.5) are that there exist a scalars λ_i^0, $i \in M$, such that

$$\nabla_x \varphi(X_0) + \sum_{i=1}^{m} \lambda_i^0 \nabla_x g_i(X_0) = 0$$

$$\sum_{i=1}^{m} \lambda_i^0 g_i(X_0) = 0$$

$$g_i(X_0) \leq 0 \qquad i \in M$$

$$\lambda_i^0 \geq 0 \qquad i \in M$$

6. Any point in P, having one of the three properties of being a stationary point, a local solution, or a global solution of the program (1.3.5), also has the other two properties.

Section III is concerned with some aspects of duality of program (1.3.5). The main result established is the Converse Duality Theorem, of Mangasarian [147] and Huard [109], generalised to more general programming problem of N.L.F.F.P.P. (1.3.5). Section IV deals with the N.L.F.F.P.P. stated as

$$\text{Optimise} \quad \varphi(X) = \frac{f(X)}{g(X)} \quad \text{for } X \in P, \tag{1.3.6}$$

where,

(i) The set P is as defined in (1.3.5c).

(ii) f is non-negative and Concave over S, g is strictly positive and concave over S. (If g be linear, then the condition of non-negativity on f can be omitted.)

(iii) The program has a finite optimum at a finite point of P.

In this section, the results analogous to results established in Section I through III of this chapter, have been stated (without proof) for the above problem (1.3.6).

In Section V we are concerned with the discussion of problems, considered by Kanti Swarup [121,123,130], Aggarwal [3], Bector [22], and the problem of L.F.F.P. It has been shown that these problems can be treated as particular cases of the more general problems considered in the previous sections of this chapter. Consequently two very interesting properties of L.F.F.P.P., viz: (i) Every local optimum is a global optimum, and (ii) a global optimum lies at an extreme point of the closed convex polyhedral set; which bear a great deal of responsibility for the development of a 'Simplex-Like' finite iteration procedure for obtaining the global optimum of L.F.F.P.P., follow as

particular cases of the results established for more general M.A.F.F.P.P.'s considered in the previous sections.

Chapter IV deals with the nature of different functions most of which occur frequently as O.F.'s and constraint functions in mathematical programming problems. For a mathematical programming problem, whether the local optimum of the O.F., on the constraint set, is global also, the Kuhn-Tucker differential conditions, which are necessary for the optimality, are sufficient also, whether the global optimum lies at one or more of the extreme points of the constraint set, depend very much on the nature of the O.F. and the constraints. An extensive literature on theoretical as well as computational aspects of linear and convex programming problems is already available, but most of the practical problems in different fields could not be represented by mathematical models of linear or convex programming. Recently, M.P.P.'s which are outside the realm of convex programming started receiving attention because one encounters such problems in diverse fields. The whole chapter is divided into five sections. First section deals with the discussion of certain properties of EQX (EQV) functions when optimised over a non-empty closed convex, if necessary assumed be bounded also. The results, (1.2.19) through (1.2.22) of convex programming, are extended to explicit quasi-concave programming. In section II, the concepts of Strong Pseudo-Convex (SPCX) functions, Weakly Convex-Like (WCXL) functions Weakly Strong Pseudo-Convex-Like (WSPCXL) functions and Quasi-Convex-Like (QXL) functions have been introduced and their relations among themselves and with other convex-like functions are established. For defining a

SPCX function we need the underlying set to be convex whereas for other (newly defined functions) we do not. The class of SPCX functions is intermediate between the class of PCX functions and CX functions. The main results proved are as follows:

(a) If a function φ is CX on a convex set S, then φ is SPCX on S, but not conversely.

(b) If φ is SPCX on S, then φ is PCX on S, but not conversely.

(c) If φ be WCXL on any set $G \subset R^n$, then it is (WSPCXL) on G, but not conversely.

(d) If φ be (WSPCXL) on G, then φ is PCX on G, but not conversely.

(e) If the convex set $S \subset G$, then a (WSPCXL) function on G is a SPCX function on S and a WCXL function is a CX function on S.

Furthermore, an alternate proof for the property that every local minimum of a PCX function, when optimised on a convex set, is global also, is provided. The main feature of the proof provided here is that it does not utilise (as used by Mangasarian [148] the property that a PCX function, on a convex set, is SQX also. In Section III, a systematic study of the nature of, product and quotient of convex-like function, rational powers of convex-like functions and composit function of convex-like functions, is made. Certain theorems of Berge [41] , and Berge and Houiri [42] proved in context with convex and quasi-convex functions, are proved, with relevant modifications, in context with SPCX, PCX and EQX functions. One of the main result proved is that a N.L.F.F. belongs to the class of SPCX (SPCV) functions, where SPCV stands for a Strong Pseudo-Concave Function. A L.F.F. is proved to be a Strong Pseudo-Monotonic (SPM) function . In Section IV, certain M.P.P.'s have been

characterized as Explicit Quasi-Convex (Concave) Programming, Pseudo-Convex (Concave) Programming, Strong Pseudo-Convex (Concave) Programming; and well known Linear Fractional Functional Programming as Strong Pseudo-Monotonic Programming. Section V deals with the characterization of some mathematical programming problems with fractional functions in the objective as convex programming problems, and some of the other outstanding achievements of this section are that it has been possible to characterize, the product of two strictly positive concave functions, and, the ratio of the product of a non-negative concave function and a strictly positive concave function to the square of a strictly positive convex functions, as EQV when at least one of the functions $\notin C^1$ and SPCV functions when all the functions $\in C^1$. The most important result of this section is that the ratio of the square of a non-negative convex function f to a strictly positive concave function g (if both f and g be linear then non-negativity restriction on f can be omitted) is a convex function.

Although the theory developed in Chapter IV is not directly concerned with the computational technique, but it has a great bearing on the development of a computational technique, typical of 'Method of Feasible Directions' [184,185] for obtaining the solution of problems of Strong Pseudo-Concave Programming and Pseudo-Concave Programming.

Chapter V deals with some computational aspects of certain type of M.P.P.'s. The chapter is divided into two major sections. Section I is devoted to the development of a computational technique, similar to 'Method of Feasible Directions' [184,185] to obtain the global maximum

of a Strong Pseudo-Concave Programming Problem, in which the O.F. is SPCV and the constraint set is a convex set constrained by non-linear QX functions which $\in C^1$, exactly on the lines of Zoutendijk, with suitable modifications for Pseudo-Concave Programming, in which the O.F. is PCV and constraint set as for Strong Pseudo-Concave Programming.

'Method of Feasible Directions' as described by Zoutendijk [184,185], is a non-finite iterative procedure for the problem of maximization of a concave function over a convex region constrained by convex functions, in which starting with a feasible solution a sequence of feasible trial solutions is generated such that the rate of increase in the O.F. at each iteration is maximum possible. To obtain a new trial solution from an old one we have to determine:

(i) a direction in which the O.F. increases and

(ii) the length of the step to be taken in that direction, such that the new trial solution remains feasible and the value of the O.F. at the new trial solution is greater than its value at the old trial solution. While initiating the procedure we must ensure that at the feasible solution with which we start, the gradient of the O.F. does not vanish, for otherwise we cannot determine a direction in which to move i.e. the iterative procedure terminates immediately. Zoutendijk [184,185] has shown that under appropriate restrictions and precautions [184,185], on the programming problem, the sequence of the trial solutions, obtained as above, can be made to converge to a local maximum and, therefore, to a global maximum when the O.F. is CV.

The essential difference between the problem considered by Zoutendijk [184,185] and the problem considered here lies in the fact that in the present problem the O.F. to be maximized is STCV(PCV) and the non-linear constraints are QX whereas in the problem considered by Zoutendijk, the O.F. is CV and the non-linear constraints are CX. But as it is well known that for a STCV(PCV) function, every local maximum over a convex set is a global maximum, therefore, we hope that the local maximum so obtained with the help of method of feasible directions will be a global maximum.

Method of feasible directions is advantageous in the sense that firstly to initiate the computations we can use any feasible solution at which the gradient of the O.F. does not vanish and it is not necessary that we should have some special form of a feasible solution (such as a B.F.S.), and secondly we may proceed into the interior of the feasible domain to increase or decrease the O.F. which usually provides a faster convergence.

Section II deals with a special type of N.L.F.F.P.P. whose mathematical moxel is as follows:

$$\text{Maximize } Z = \frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta} + \left(\frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right)^3 + \cdots + \left(\frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right)^{2p+1}$$

subject to

$$\sum_{j=1}^{n} a_{ij} x_j \leq b_i \qquad (i = 1, 2, ---, m)$$

$$x_j \geq 0 \qquad (j = 1, 2, ---, n)$$

where p is a non-negative finite integer, and the following assumptions hold.

(i) All a_{ij}, b_i, c_j, d_j, α and β are known constants.

(ii) The set of feasible solutions is regular.

(iii) $\sum_{j=1}^{n} d_j x_j + \beta > 0$ for all feasible solutions.

The objective function is proved to be PCM which ensures that, (1) every local maximum of O.F. is global also and, (2) A global maximum is achieved at one of the extreme points of the constraint set. This leads to the development of a finite iteration technique similar to that developed by Kanti Swarup [122] for L.F.P. which is typical of Beale's Method for Quadratic Programming [16]. Sufficient conditions for optimality are obtained.

CHAPTER - II

NON-LINEAR INDEFINITE FUNCTIONAL PROGRAMMING

INTRODUCTION:

The purpose of this chapter is to deal with a variety of topics related with certain theoretical aspects of fundamental nature for the problem of N.L.I.F.P. in which the O.F. which is neither convex nor concave but is more general than a concave function, is the product of two non-linear strictly positive (assumed to be differentiable if and when needed) concave functions, and the constraint set is a convex set (assumed to be constrained by specified (more general than convex or concave) functions if and when needed). The chapter is divided into four sections. The results established in Section I are mainly that for such a problem, (i) every local maximum is global maximum, (ii) the set of points, at which the global maximum is taken on, is a convex set,

(iii) if the O.F. has a point of global minimum in the interior of the set of feasible solutions, then the O.F. is constant throughout the constraint set, (iv) if the constraint set be compact also, then the global minimum of the O.F. will be taken on at one or more of the extreme points of the constraint set. In Section II, the Kuhn-Tucker differential conditions for the existence of a local maximum (which is global maximum also), have been proved to be necessary as well as sufficient. Section III is devoted to certain aspects of duality. In Section IV a few particular cases considered by the author and other research workers have been discussed.

[illegible]:

Below we now give a few results, in the form of lemmas, which we shall use in the analysis to follow .

LEMMA 1. Every compact non-empty convex set admits at least one extreme point [41,42] .

Lemma 2. Let H_{n-1} be a supporting hyperplane of a non-empty compact convex set $S_n \subset R^n$. Then the set $S_{n-1} = H_{n-1} \cap S_n$ is a compact non-empty convex set in (n-1)-dimensional space [41] .

Lemma 3. If u_i, $i=1,2,\text{------},p$ be strictly positive real numbers, then

$$\psi(u_1, u_2, ----, u_p) = u_1 + u_2 + ---- + u_p - \log u_1 - \log u_2 - ---- - \log u_p - p$$

where p is a finite positive integer, is always non-negative.

Proof. To prove $\psi(u_1, u_2, \text{------}, u_p) \geqslant 0$, it suffices to prove that $\underset{u_1, u_2, ---, u_p}{\text{minimum}} \psi(u_1, u_2, \text{------}, u_p) = 0$ for all u_i strictly positive and in R. To find min ψ we proceed as follows.

To obtain its stationary points we set

$$\frac{\partial \psi}{\partial u_i} = 1 - \frac{1}{u_i} = 0 \quad \text{for} \quad i = 1, 2, \cdots\cdots, p$$

which imply that $(u_1, u_2, \text{------}, u_p) = (1, 1, \text{------}, 1)$ is the only stationary point of $\psi(u_1, u_2, \text{------}, u_p)$. To find whether the function takes on its maximum or minimum at $(1, 1, \text{------}, 1)$ we find that

$$\frac{\partial^2 \psi}{\partial u_i^2} = \frac{1}{u_i^2} = 1 \quad \text{at} \quad (1, 1, \cdots\cdots, 1) \quad \text{for} \quad i = 1, 2, \cdots\cdots, p$$

$$\frac{\partial^2 \psi}{\partial u_i \partial u_j} = 0 = \frac{\partial^2 \psi}{\partial u_j \partial u_i} \qquad i, j = 1, 2, \cdots\cdots, p \;;\; i \neq j$$

and the matrix

$$\begin{bmatrix} \frac{\partial^2 \psi}{\partial u_1^2} & \frac{\partial^2 \psi}{\partial u_1 \partial u_2} & \cdots\cdots & \frac{\partial^2 \psi}{\partial u_1 \partial u_p} \\ \frac{\partial^2 \psi}{\partial u_2 \partial u_1} & \frac{\partial^2 \psi}{\partial u_2^2} & \cdots\cdots & \frac{\partial^2 \psi}{\partial u_2 \partial u_p} \\ \vdots & \vdots & & \vdots \\ \frac{\partial^2 \psi}{\partial u_p \partial u_1} & \frac{\partial^2 \psi}{\partial u_p \partial u_2} & \cdots & \frac{\partial^2 \psi}{\partial u_p^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots\cdots & 0 \\ 0 & 1 & \cdots\cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots\cdots & 1 \end{bmatrix}$$

is positive-definite, which implies that $\psi(u_1, u_2, \text{------}, u_p)$ takes on its minimum at $(1, 1, \text{------}, 1)$ and that $\underset{u_1, u_2, \cdots, u_p}{\text{minimum}} \; \psi\,(u_1, u_2, \text{------}, u_p) = 0.$

Hence the result follows.

SECTION - I

THEORETICAL RESULTS OF FUNDAMENTAL NATURE:

Our primary concern in this section is with the N.L.I.F.P.P. stated as follows:

Optimise (maximise or minimise) $\varphi(X) = f(X)\, g(X)$ for $X \in P$. (2.1.1)

where,

(i) $P \subset S$ is a non-empty closed convex set containing at least two points, and

(ii) f, g are functions which are CV over S and strictly positive throughout P, so that the O.F. φ is neither CX nor CV over P.

(iii) A finite optimum occurs at a finite point of P.

The main purpose of this section is to exhibit that for such a problem given by (2.1.1), in which the O.F. is neither CX nor CV, the results of 'Convex Programming' extend as follows:

(a) Every local maximum of φ over P is also a global maximum over P.

(b) The set of points in P at which φ takes on its global maxima is a convex set.

(c) If the O.F. φ has a point of global minimum in the interior of the set P, then φ is constant throughout P.

(d) If the constraint set P be bounded also, then the global minimum of over P will be taken on at one or more of the extreme point of P.

Theorem 1. Every local maximum of φ over P is also a global maximum of φ over P.

Proof. If possible let the theorem be false, such that if $X_0 \in P$ is a local maximum of φ over P and $X_* \; (\neq X_0) \in P$ is a global maximum, then we have

$$\varphi(X_*) > \varphi(X_0) \tag{2.1.2}$$

Consider

$$X_\lambda = \lambda X_* + (1-\lambda) X_0 \in P \quad \text{for all } \lambda \in [0,1]. \tag{2.1.3}$$

of λ for $0 < \lambda < 1$, it is possible for us to choose X_λ in such a way that, $X_\lambda \in \mathcal{N}_\varepsilon(X_0) \cap P$, and we thus have

$$\varphi(X_\lambda) \leq \varphi(X_0)$$

which shows that (2.1.5) is a contradiction. Thus the result follows.

Theorem 2. The set of those points in P at which φ assumes its global maximum over P, is a convex set.

Proof. Let M (a sub-set of P) be the set of points at each of which φ takes on its global maximum.

If M is empty or contains just a single point then it is evidently a convex set. If M contains more than one point, then let X_1 and X_2 be two different points in M, at which φ takes on its global maximum over P. Then we have

$$\varphi(X_1) = \varphi(X_2) \qquad \text{for} \quad X_1, X_2 \in M \subset P. \tag{2.1.6}$$

Now for all $\lambda \in [0, 1]$, we have,

$$\varphi[\lambda X_1 + (1-\lambda) X_2] = f[\lambda X_1 + (1-\lambda) X_2]\, g[\lambda X_1 + (1-\lambda) X_2] \tag{2.1.7}$$

Exactly as in Theorem I, (2.1.6) and (2.1.7) yield,

$$\varphi[\lambda X_1 + (1-\lambda) X_2] \geq \varphi(X_1) = \varphi(X_2) \quad \text{for all } \lambda \in]0,1[.$$

But $\varphi[\lambda X_1 + (1-\lambda) X_2]$ cannot be greater than $\varphi(X_1) = \varphi(X_2)$ as both X_1 and X_2 are points of global maximum of φ over P. Thus we have $\forall \lambda \in]0,1[$

$$\varphi[\lambda X_1 + (1-\lambda) X_2] = \varphi(X_1) = \varphi(X_2)$$

This implies that $\lambda X_1 + (1-\lambda) X_2 \in M$ for all $\lambda \in [0,1]$ i.e. the set of points at which φ takes on its global maximum is a convex set.

Now we state a few corollaries which are either the immediate consequences of the above results, or else, can be proved easily.

Corollary 1. If φ takes on its global maximum at two different points in P, then it takes on its global maximum at an infinite number of points in P.

Corollary 2. There cannot be two (or more) points in P at which φ takes on a strong local (and hence a global) maximum.

Corollary 3. If at least one of the functions f and g is strictly concave, then the global maximum of φ over the set P is taken on at a unique point.

Theorem 3. If φ has a global minimising point in the interior of the constraint set P, then φ is constant over P.

Proof. Let φ be not constant over P and let X_* be an interior point of the set P, such that

$$\varphi(X_*) = \min_{X \in P} \varphi(X) \tag{2.1.8}$$

Again, φ being not constant over P, we have a point X_1 (different from X_*) $\in$ P, such that

$$\varphi(X_1) > \varphi(X_*) \tag{2.1.9}$$

Let $X_2 \in P$ be another point different from both X_1 and X_* and that

$$X_* = \lambda X_1 + (1-\lambda) X_2 \qquad \forall \lambda \in]0,1[. \tag{2.1.10}$$

Now because $X_2 \in P$, therefore we may have either $\varphi(X_2) = \varphi(X_*)$ or $\varphi(X_2) > \varphi(X_*)$. In the former case with the help of (2.1.9) we infer that we have, (1) $\varphi(X_1) > \varphi(X_2)$, whereas in the latter case we may have, (2) either (a) $\varphi(X_1) > \varphi(X_2)$ or (b) $\varphi(X_1) < \varphi(X_2)$ or (c) $\varphi(X_1) = \varphi(X_2)$. Thus we can have in all any and only one of the following three possibilities.

P-1. $\varphi(X_1) > \varphi(X_2)$; when either $\left.\begin{array}{l}\text{(i) } \varphi(X_2) > \varphi(X_*) \\ \text{or (ii) } \varphi(X_2) = \varphi(X_*)\end{array}\right]$ (2.1.11)

P-2. $\varphi(X_1) < \varphi(X_2)$; when $\varphi(X_2) > \varphi(X_*)$ (2.1.12)

P-3. $\varphi(X_1) = \varphi(X_2)$; when $\varphi(X_2) > \varphi(X_*)$ (2.1.13)

Thus we have for all $\lambda \in]0,1[$,

$$\varphi(X_*) = f[\lambda X_1 + (1-\lambda) X_2]\, g[\lambda X_1 + (1-\lambda) X_2] \tag{2.1.14}$$

Making use of the concavity and strict positivity of both f and g over P, from (2.1.14) we obtain,

$$\varphi(X_*) \geq [\lambda f(X_1) + (1-\lambda) f(X_2)][\lambda g(X_1) + (1-\lambda) g(X_2)]$$

$$= \lambda^2 \varphi(X_1) + (1-\lambda)^2 \varphi(X_2) + \lambda(1-\lambda)[f(X_1) g(X_2) + f(X_2) g(X_1)] \tag{2.1.15}$$

Assuming that possibility P-1 exists and proceeding exactly as in Theorem 1, (2.1.15) in conjunction with (2.1.11) yields

$$\varphi(X_*) > \varphi(X_2) \tag{2.1.16}$$

Similarly if we assume that either possibility P-2 or possibility P-3 exists, we will obtain,

either $\varphi(X_*) > \varphi(X_1)$ (2.1.17)

or $\varphi(X_*) \geqslant \varphi(X_1) = \varphi(X_2) > \varphi(X_*)$ (2.1.18)

respectively.

But results (2.1.16) through (2.1.18) are contradictions. Hence the result follows.

Theorem 4. If the non-empty convex set P be bounded also then φ takes on its global minimum over P at one or more of extreme points of P.

Proof. Since the set P is a non-empty compact convex set, therefore, by Lemma 1, it admits at least one extreme point. Now when φ attains its global minimum in the interior of the set P, then by Theorem 3, is constant over P and, therefore, takes on its global minimum at an extreme point of P.

Let us now assume that φ takes on its global minimum over P at a boundary point $X_* \in P$ and not at any interior point of P. In case X_* is already an extreme point of P, we have proved the result. If not, then we consider the set $S_{n-1} = H^*_{n-1} \cap P$, where H^*_{n-1} is a supporting hyperplane of P at point $X_* \in P$. Since the set P is a non-empty compact convex set with $P \subset S \subset E^n$, therefore by Lemma 2, S_{n-1} is also a non-empty compact convex set lying in (n-1)-dimensional space. Thus Lemma 1 implies that S_{n-1} admits at least one extreme point. Now if X_* happens to be an interior point of S_{n-1} with respect to the space of dimensions (n-1), then by Theorem 3 we immediately conclude that φ will

be constant over S_{n-1} and therefore it attains its global minimum at every point of S_{n-1} and hence at an extreme point of S_{n-1}, which will also be an extreme point of P. If X_* is on the boundary, we obtain another non-empty compact convex set $S_{n-2} = H^*_{n-2} \cap S_{n-1}$ with S_{n-2} lying in (n-2)-dimensional space, where H^*_{n-2} is a supporting hyperplane of the set S_{n-1} at point X_*, and repeat the above argument. This process must terminate in a finite number of stages because S_0 will be a non-empty convex set lying in 0-dimensional space and hence will contain only the single point X_*, which by Lemma 1 must be the extreme point of S_0 and therefore of P. Hence the result.

Remark 1. Although we have established that the O.F. φ in the N.L.I.F.P.P. (2.1.1) will attain its global minimum at one of the extreme points of P, yet it is not possible for us to exploit this fact to develop a computational technique, similar to Simplex Method, to find the global minimum of φ, since the well known, "Adjacent Extreme Point Methods," do not necessarily yield the global minimum and in our case it is possible that φ may take on its local minimum, different from global minimum at an extreme point of the constraint set P. However, to obtain a local minimum for such a problem, a 'Simplex-Like Technique' can be developed.

SECTION - II

KUHN-TUCKER OPTIMALITY CONDITIONS:

In Section I, Theorem 1, we have proved for the N.L.I.F.P.P. given by (2.1.1) an interesting property that every local maximum of over P is global maximum also. In the present section we propose to establish the necessary and sufficient condition, similar to those given

by Kuhn and Tucker [142] , for the existence of a local maximum (and hence a global maximum) of the Problem (2.1.1), with the additional assumption that the set P is given by

$$P = \left\{ X \; ; \; g_i(X) \leq 0 \; , \; i \in M = (1, 2, \cdots, m) ; \; X \in S \right\} \qquad (2.2.1)$$

where all the functions g_i in the system

$$g_i(X) \leq 0 \qquad i \in M \qquad (2.2.2)$$

are QX such that P is a closed convex set.

We now prove the following theorems.

Theorem 1. If $X_0 \in P$ (2.2.1) be a point which is sequentially qualified, [11], for the system (2.2.2) and if all the functions f, g and g_i, $i \in M$; $\in C^1$ at X_0, then the necessary and sufficient conditions that X_0 be a maximum to the program given by (2.1.1), (2.2.2) and (2.2.2) are that there exist m scalars λ_i^0 , $i \in M$, such that

$$\nabla_x \varphi(X_0) = \sum_{i=1}^{m} \lambda_i^0 \nabla_x g_i(X_0) \qquad (2.2.3)$$

$$\sum_{i=1}^{m} \lambda_i^0 \, g_i(X_0) = 0 \qquad (2.2.4)$$

$$g_i(X_0) \leq 0 \qquad i \in M \qquad (2.2.5)$$

$$\lambda_i^0 \geq 0 \qquad i \in M \qquad (2.2.6)$$

Proof. To show that the conditions (2.2.3) through (2.2.6) are necessary we give a proof similar to that of [11, Theorem 4] .

Let

$$N_a^0 = \left\{ i \; ; \; g_i(X_0) = 0 \right\} \qquad (2.2.7)$$

$$N_I^0 = \left\{ i \; ; \; g_i(X_0) < 0 \right\} \qquad (2.2.8)$$

where $\qquad N_a^o = M - N_I^o$

Then, from (2.2.4) through (2.2.8) we obtain that $\lambda_i^o = 0$ for $i \in N_I^o$.
If possible let the system (2.2.3), (2.2.4), (2.2.6) have no solution, this implies that

$$\nabla_x \varphi(x_o) = \sum_{i \in N_a^o} \lambda_i^o \nabla_x g_i(x_o) \tag{2.2.9}$$

$$\lambda_i^o \geq 0 \qquad i \in N_a^o \tag{.2.10}$$

have no solution. By the Transposition Theorem (see Chapter I) and from the fact that (2.2.9) and (2.2.10) have no solution it follows that there exists some vector $Y \in R^n$ such that

$$[\nabla_x \varphi(x_o)]' Y > 0 \tag{2.2.11}$$

$$[\nabla_x g_i(x_o)]' Y \leq 0 \qquad i = 1, 2, \cdots, m \tag{2.2.12}$$

(2.2.12) yields that Y belongs to linearising cone to (2.2.2) at $X_o \in P$. Since X_o is sequentially qualified, this implies that Y belongs to the cone of tangents to P at X_o. Therefore, there exists some sequence (X_p) contained in P and converging to X_o, and some sequence (λ_p) of non-negative numbers, such that the sequence $((X_p - X_o)\lambda_p)$ converges to Y. Again we have,

$$\varphi(x_p) = \varphi(x_o) + [\nabla_x \varphi(x_o)]'(x_p - x_o) + \|x_p - x_o\| \varepsilon \tag{2.2.13}$$

where ε is a scalar, depending on p, and tending to zero as p tends to infinity, and, $\| \quad \|$, denotes the norm. From (2.2.13), we have

$$[\varphi(x_p) - \varphi(x_o)]\lambda_p = [\nabla_x \varphi(x_o)]'(x_p - x_o)\lambda_p + \|(x_p - x_o)\lambda_p\| \varepsilon \tag{2.2.14}$$

Now if we make p sufficiently large such that $\varepsilon \to 0$ and $(X_p - X_0)\lambda_p$ converges to Y, then $\|(X_p - X_0)\lambda_p\| \varepsilon \to \|Y\| \varepsilon$ with $\varepsilon \to 0$, which implies that the right hand side in (2.2.14) has the sign of $[\nabla_x \varphi(X_0)]' Y$ which by (2.2.11) is positive. This implies that

$$\varphi(X_p) > \varphi(X_0) \tag{2.2.15}$$

Since $X_p \in P$, therefore, (2.2.15) is a contradiction to the fact that X_0 is a maximum. Hence the result follows.

To prove that the conditions (2.2.3) through (2.2.6) are sufficient also, we assume that there exist X_0 and λ_i^0, $i \in M$ satisfying those conditions.

We now have for X_0 and any $X \in P$

$$(X - X_0)' \nabla_x \varphi(X_0) = g(X_0)(X - X_0)' \nabla_x f(X_0) + f(X_0)(X - X_0)' \nabla_x g(X_0) \tag{2.2.16}$$

Using the concavity and strict positivity of f and g, we obtain from (2.2.16)

$$(X - X_0)' \nabla_x \varphi(X_0) \geqslant g(X_0)\left[f(X) - f(X_0)\right] + f(X_0)\left[g(X) - g(X_0)\right]$$

$$= \varphi(X_0)\left[u_1 + u_2 - 2\right]$$

where $\quad u_1 = \dfrac{f(X)}{f(X_0)}, \quad u_2 = \dfrac{g(X)}{g(X_0)}$

$$\Rightarrow (X - X_0)' \nabla_x \varphi(X_0) \geqslant \varphi(X_0)\left[\log \varphi(X) - \log \varphi(X_0)\right]$$
$$+ \varphi(X_0)\left[u_1 + u_2 - \log u_1 - \log u_2 - 2\right]$$
$$\geqslant \varphi(X_0)\left[\log \varphi(X) - \log \varphi(X_0)\right]$$

$$\Rightarrow \quad \log \varphi(X) - \log \varphi(X_0) \leqslant \frac{1}{\varphi(X_0)}\left[(X - X_0)' \nabla_x \varphi(X_0)\right] \tag{2.2.17}$$

Now if we make p sufficiently large such that $\varepsilon \to 0$ and $(X_p - X_0)\lambda_p$ converges to Y, then $\|(X_p - X_0)\lambda_p\|\varepsilon \to \|Y\|\varepsilon$ with $\varepsilon \to 0$, which implies that the right hand side in (2.2.14) has the sign of $[\nabla_x \varphi(X_0)]' Y$ which by (2.2.11) is positive. This implies that

$$\varphi(X_p) > \varphi(X_0) \tag{2.2.15}$$

Since $X_p \in P$, therefore, (2.2.15) is a contradiction to the fact that X_0 is a maximum. Hence the result follows.

To prove that the conditions (2.2.3) through (2.2.6) are sufficient also, we assume that there exist X_0 and λ_i^0, $i \in M$ satisfying those conditions.

We now have for X_0 and any $X \in P$

$$(X - X_0)' \nabla_x \varphi(X_0) = g(X_0)(X - X_0)' \nabla_x f(X_0) + f(X_0)(X - X_0)' \nabla_x g(X_0) \tag{2.2.16}$$

Using the concavity and strict positivity of f and g, we obtain from (2.2.16)

$$(X - X_0)' \nabla_x \varphi(X_0) \geq g(X_0)\left[f(X) - f(X_0)\right] + f(X_0)\left[g(X) - g(X_0)\right]$$

$$= \varphi(X_0)\left[u_1 + u_2 - 2\right]$$

where $\quad u_1 = \dfrac{f(X)}{f(X_0)}, \quad u_2 = \dfrac{g(X)}{g(X_0)}$

$$\Rightarrow (X - X_0)' \nabla_x \varphi(X_0) \geq \varphi(X_0)\left[\log \varphi(X) - \log \varphi(X_0)\right] + \varphi(X_0)\left[u_1 + u_2 - \log u_1 - \log u_2 - 2\right]$$

$$\geq \varphi(X_0)\left[\log \varphi(X) - \log \varphi(X_0)\right]$$

$$\Rightarrow \quad \log \varphi(X) - \log \varphi(X_0) \leq \frac{1}{\varphi(X_0)}\left[(X - X_0)' \nabla_x \varphi(X_0)\right] \tag{2.2.17}$$

From (2.2.3) we have

$$(x-x_0)'\nabla_x \varphi(x_0) = \sum_{i=1}^{m} \lambda_i^0 (x-x_0)' \nabla_x g_i(x_0) \tag{2.2.18}$$

Again from (2.2.4) through (2.2.6) and (2.2.7), (2.2.8) we already have,

$$\lambda_i^0 = 0 \qquad \text{for} \qquad i \in N_I^0 . \tag{2.2.19}$$

Using (2.2.1) and (2.2.7) for $X \in P$, we have

$$g_i(X) \leq g_i(X_0) \qquad i \in N_\alpha^0 \tag{2.2.20}$$

In view of quasi-convexity of the functions g_i's, $i \in M$, we have

$$g_i(x) \leq g_i(x_0) \Rightarrow (x-x_0)'\nabla_x g_i(x_0) \leq 0 \tag{2.2.21}$$

Thus (2.2.20) and (2.2.21) yield,

$$(x-x_0)'\nabla_x g_i(x_0) \leq 0 \quad \text{for} \quad i \in N_\alpha^0 \quad \text{and} \quad x \in P. \tag{2.2.22}$$

From (2.2.6) and (2.2.22) we get,

$$\sum_{i \in N_\alpha^0} \lambda_i^0 (x-x_0)' \nabla_x g_i(x_0) \leq 0 \quad \text{for} \quad x \in P. \tag{2.2.23}$$

Using (2.2.19) we have,

$$\sum_{i \in N_I^0} \lambda_i^0 (x-x_0)' \nabla_x g_i(x_0) = 0 \quad \text{for} \quad x \in P. \tag{2.2.24}$$

Combining (2.2.23) and (2.2.24) we get,

$$\sum_{i=1}^{m} \lambda_i^0 (x-x_0)' \nabla_x g(x_0) = 0 \quad \text{for} \quad x \in P \tag{2.2.25}$$

From (2.2.17), (2.2.18) and (2.2.25) we obtain,

$$\log \varphi(x) \leq \log \varphi(x_0) \quad \text{for} \quad x \in P$$

$$\Rightarrow \quad \varphi(x) \leq \varphi(x_0) \quad \text{for} \quad x \in P$$

This proves that φ takes on its maximum at X_0.

Remark 1. In above either N_I° or N_a° can happen to be a null set. To modify the proof, in the former case we omit (2.2.19), (2.2.24) and the associated references and in the latter case we omit (2.2.20), (2.2.22) and (2.2.23).

Corollary 1. If for $X_0 \in P$, $\nabla_x \varphi(X_0) = 0$, then φ takes on its global maximum over P, at X_0.

Proof. We have $\nabla_x \varphi(X_0) = 0$ for $X_0 \in P$

Therefore, for all $X \in P$, $(X - X_0)' \nabla_x \varphi(X_0) = 0$ for $X_0 \in P$ (2.2.26)

Using (2.2.17) and (2.2.26) we obtain

$$\log \varphi(X) \leqslant \log \varphi(X_0) \qquad \text{for } X_0 \text{ and all } X \in P.$$

$$\Rightarrow \varphi(X) \leqslant \varphi(X_0) \qquad \text{for } X_0 \text{ and all } X \in P.$$

Hence the result.

Wolfe [183] has proved that if ψ be a concave function on a convex set S, then any point having one of the three properties of being a stationary point, a local solution, or a global solution, also has the other two properties. Below we prove the theorem for a more general function φ given in (2.2.1).

Theorem 2. For the function $\varphi = fg$ given by (2.2.1), and satisfying the assumptions stated in Sections I and II, any point in P having one of the three properties of being a stationary point, a local solution, or a global solution, also has the other two properties.

Proof. The result follows immediately on making use Theorem 1 of Section I, Theorem 1 and Corollary 1 to Theorem 1 of Section II.

SECTION - III

SOME ASPECTS OF DUALITY

We shall devote this section to discuss certain aspects of duality of the N.L.I.F.P.P. considered in this chapter. It is well known that the duality principles in convex programming connect two programming problems, one of which, called the (PP) is a constrained maximization (or minimization) and the other one called (DP) is a constrained minimization (or maximization) problem. For the sake of convenience we state the primal and dual problems as follows:

PRIMAL PROBLEM (PP)

$$\text{Maximize} \quad \varphi(X) = f(X)\,g(X) \quad \text{for} \quad X \in P. \tag{2.3.1}$$

where

(i) the set $P \subset S \subset R^n$ is defined as follows:

$$P = \left\{ X \,;\, g_i(X) \leqslant 0,\ i \in M = (1, 2, \cdots, m) \,;\, X \in S \right\} \tag{2.3.2}$$

it being assumed that the functions g_i, for all $i \in M$, are CX such that the set P is a closed convex set.

(ii) $X_0 \in P$ is the optimal solution of (2.3.1) and is sequentially qualified for the system $g_i(X) \leqslant 0$, $i \in M$.

(iii) functions g_i for all $i \in M$ and $f, g \in C^2$ at X_0.

(iv) f, g are concave over S and strictly positive over P such that the O.F. φ is strictly positive and neither CX nor CV over P.

DUAL PROBLEM (DP):

For the problem (2.3.1), according to Wolfe [181] the (DP) is:

$$\text{Minimize} \quad F(x,\lambda) = \varphi(x) - \sum_{i=1}^{m} \lambda_i g_i(x) \quad \text{for } (x,\lambda) \in D \quad (2.3.3)$$

where,

(i) the set $D \subset R^{n+m}$ is given by,

$$D = \left\{ (x,\lambda) \; ; \; \nabla_x \varphi(x) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x) \; ; \; \lambda \geq 0 \right\} \quad (2.3.4)$$

(ii) $\lambda = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_m)'$ is an mx1 column vector $\in R^m$

(iii) (X_0, λ_0) D is sequentially qualified for the system

$$\nabla_x \varphi(x) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x)$$

Wolfe [181], under appropriate restrictions, has shown for the pair of problems (1.2.45), (1.2.46) that if X_0 solves (PP) then X_0 and some λ_0 solves (PP). Conversely, under somewhat stronger conditions, Huard [109] and Mangasarian [147], have shown that (X_0, λ_0) solves (DP) then X_0 solves (PP). Wolfe [181], Huard [109], and Mangasarian [147], among other things require that the O.F. be concave and the constraint functions be convex. In the present case it is shown that Mangasarian [147] and Huard [109] Theorems can be extended to more general case of N.L.I.F.P.

Before stating the main theorems of this section, we first state the following well known result in the form of a lemma, which we shall use in Theorem 2 below.

Lemma 1. If B be an nxn order, non-singular square matrix, then the homogeneous system of linear equations in n unknowns, given by $BX = 0$, $X \in R^n$, has only the trivial solution $X = 0$.

We now prove the following theorems.

Theorem 1. If X_0 maximises (PP), then there exists a point $(X_0, \lambda_0) \in D$ such that $F(X_0, \lambda_0) = \max\limits_{X \in P} \varphi(X)$ and $\min\limits_{\substack{(X_0,\lambda)\in D \\ X_0 \in P}} F(X_0, \lambda) \geq F(X_0, \lambda_0)$ for all $(X_0, \lambda) \in D$.

Proof. It is given that X_0 maximises the (PP), therefore, by Theorem 1, Section II, we have that corresponding to X_0, it is necessary and sufficient that there exists a vector $\lambda_0 = (\lambda_1^0, \lambda_2^0, \text{-----}, \lambda_m^0)'$ such that

$$\nabla_x \varphi(X_0) = \sum_{i=1}^{m} \lambda_i^0 \nabla_x g_i(X_0) \tag{2.3.5}$$

$$\sum_{i=1}^{m} \lambda_i^0 g_i(X_0) = 0 \tag{2.3.6}$$

$$g_i(X_0) \leq 0 \qquad i \in M \tag{2.3.7}$$

$$\lambda_i^0 \geq 0 \qquad i \in M \tag{2.3.8}$$

From (2.3.5) and (2.3.8) we get that there exists $(X_0, \lambda_0) \in D$.

Again we have

$$F(X_0, \lambda_0) = \varphi(X_0) - \sum_{i=1}^{m} \lambda_i^0 g_i(X_0) \tag{2.3.9}$$

(2.3.6) and (2.3.9) yield

$$F(X_0, \lambda_0) = \varphi(X_0) = \max_{X \in P} \varphi(X)$$

This proves the first desired result.

Now we have that $F(X, \lambda)$ is a linear function in λ such that (2.3.6) is satisfied for (2.3.7). Therefore, for $\lambda \geqslant 0$ we have

$$\sum_{i=1}^{m} \lambda_i g_i(X_0) \leqslant 0 = \sum_{i=1}^{m} \lambda_i^0 g_i(X_0)$$

This implies $$-\sum_{i=1}^{m} \lambda_i g_i(X_0) \geqslant -\sum_{i=1}^{m} \lambda_i^0 g_i(X_0) \qquad \forall \lambda_i \geqslant 0$$

i.e. $$\min_{\lambda_i \geqslant 0} \left[-\sum_{i=1}^{m} \lambda_i g_i(X_0)\right] \geqq -\sum_{i=1}^{m} \lambda_i^0 g_i(X_0)$$

Now $$\min_{\substack{(X_0,\lambda)\in D \\ X_0 \in P}} F(X_0,\lambda) = \min_{\substack{(X_0,\lambda)\in D \\ X_0 \in P}} \left[\varphi(X_0) - \sum_{i=1}^{m} \lambda_i g_i(X_0)\right]$$

$$\geqq F(X_0, \lambda_0)$$

This proves the second result.

Theorem 2 (The Converse Duality Theorem) : If (X_0, λ_0) is an optimal solution to (DP) and $F(X, \lambda_0)$ is twice continuously differentiable with respect to X in a neighbourhood of X_0, and if the Hessian of $F(X, \lambda_0)$ with respect to X is non-singular at X_0, then X_0 optimizes (PP).

Proof. We define the function

$$H(X,\lambda,V) \equiv \varphi(X) - \lambda' G(X) + V'\nabla_X \varphi(X) - V'\nabla_X \lambda' G(X)$$

where, (i) $G(X) = [g_1(X), g_2(X), \text{------}, g_m(X)]'$ is an mx1 vector function such that $\lambda' G(X) = \sum_{i=1}^{m} \lambda_i g_i(X)$, and (ii) V is an nx1 column vector of variables.

Since (X_o, λ_o) is sequentially qualified for the system

$\nabla_x \varphi(X) = \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X)$, therefore, the necessary conditions for (X_o, λ_o) to be a solution of the (DP) are, as given by [11, 142], that there must exist a vector $V_o \in R^n$, such that,

$$\nabla_x \varphi(X_o) - \nabla_x \lambda_o' G(X_o) + \nabla_x V_o' \nabla_x \varphi(X_o) - \nabla_x V_o' \nabla_x \lambda_o' G(X_o) = 0 \qquad (2.3.10)$$

$$G(X_o) + \nabla_\lambda V_o' \nabla_x \lambda_o' G(X_o) \leq 0 \qquad (2.3.11)$$

$$\nabla_x \varphi(X_o) - \nabla_x \lambda_o' G(X_o) = 0 \qquad (2.3.12)$$

$$\lambda_o' G(X_o) - \lambda_o' \nabla_\lambda V_o' \nabla_x \lambda_o' G(X_o) = 0 \qquad (2.3.13)$$

$$\lambda_o \geqslant 0 \qquad (2.3.14)$$

(2.3.10) and (2.3.12) taken together imply:

$$\nabla_x V_o' \nabla_x \varphi(X_o) - \nabla_x V_o' \nabla_x \lambda_o' G(X_o) = 0$$

$$\Longrightarrow \quad \nabla_x V_o' \nabla_x F(X_o, \lambda_o) = 0 \qquad (2.3.15)$$

Further (2.3.15) in conjunction with the assumption that the Hessian of $F(X, \lambda_o)$ at X_o is non-singular and Lemma 1 of this section yield,

$$V_o = 0 \qquad (2.3.16)$$

Combining (2.3.10) and (2.3.16) we obtain,

$$\nabla_x \varphi(X_o) = \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(X_o) \qquad (2.3.17)$$

(2.3.13) together with (2.3.16) gives,

$$\sum_{i=1}^{m} \lambda_i^{\circ} g_i(x_\circ) = 0 \qquad (2.3.18)$$

Using (2.3.16) in (2.3.11) we get

$$g_i(X_\circ) \leq 0 \qquad i \in M \qquad (2.3.19)$$

From Section II, Theorem 1, we infer that the conditions (2.3.17) through (2.3.19) together with (2.3.14) are no other than the necessary and sufficient conditions for X_o to be the maximum of (PP). This proves the theorem.

SECTION - IV

PARTICULAR CASES: Below we now give some problems discussed by the author and other research workers, which can be considered as particular cases of N.L.I.F.P.P. considered in this chapter.

Problem 1. Indefinite Quadratic Programming (I.Q.P.) with Linear Constraints:

Kanti Swarup [124,125,126] considered the following I.Q.P. problem

Maximise $\varphi(X) = (c'X + \alpha)(d'X + \beta)$ (2.4.1a)

subject to

$$\left.\begin{array}{l} AX \leq b \\ X \geq 0 \end{array}\right\} \qquad (2.4.1b)$$

where the symbols have their meanings as in Chapter I and also the assumptions of (1.2.34), Chapter I, hold. He reduced the I.Q.P. problem to convex programming problem. Since we know that every linear function is both convex and concave and every continuous

convex function on a convex set is quasi-convex also, therefore, we conclude that I.Q.P. problem given by (2...1) is a particular case of the problem considered in this chapter. Hence all the results established in Sections I through Section III hold for I.Q.P. problem.

Problem 2. Indefinite Quadratic Programming (I.Q.P.) with non-Linear Constraints:

In [131] Kanti Swarup considered the following I.Q.P. problem

Maximize $$\varphi(X) = (c'X+\alpha)(d'X+\beta) \tag{2.4.2a}$$

subject to

$$g_i(X) \leqslant 0 \quad , \quad i \in M \tag{2.4.2b}$$

where, $g_i(X)$ for all $i \in M$ are CX functions assumed to be differentiable and satisfying Kuhn-Tucker constraint qualification [142] , the other assumptions being same as for Problem 1 above.

Since every CX function defined on a convex set is QX also, therefore, problem (2.4.2) is a particular case of the problem of N.L.I.F.P. Therefore, the results established for N.L.I.F.P.P. in Sections I through III hold for (2.4.2).

Problem 2. Indefinite Quadratic Programming (I.Q.P.) with Standard Errors in Objective:

Bector [24] considered the following I.Q.P. problem

Maximise $$\varphi(X) = \left[c'X - (X'HX)^{1/2} + \alpha\right]\left[d'X+\beta\right] \tag{2.4.3a}$$

subject to

$$\left.\begin{array}{l} AX \leqslant b \\ X \geqslant 0 \end{array}\right\} \tag{2.4.3b}$$

where,

(i) H is an nxn real symmetric positive semi-definite matrix

(ii) $C'X-(X'HX)^{\frac{1}{2}}+\alpha$, $d'X+\beta$ are finite and strictly positive over the set of feasible solutions (2.4.3b), and

(iii) rest of the symbols have the same meanings as in problem (2.4.2).

Bector 24 reduced the problem to a convex programming problem from which the solution of (2.4.3) could be obtained. Here if we assume the matrix H to be positive definite such that, over the set (2.4.3b), $C'X-(X'HX)^{\frac{1}{2}}+\alpha$ is differentiable also, then we see that the problem (2.4.3) is a particular case of the N.L.I.F.P.P. considered in this chapter.

Problem 4. Indefinite Quartic Programming with Standard Errors in Objective:

Bector [25] considered the following N.L.I.F.P.P.

$$\text{Maximise } \varphi(X) = \left[C'X - X'EX - (X'GX)^{1/2} + \alpha\right]\left[d'X - X'FX - (X'HX)^{1/2} + \beta\right] \qquad (2.4.4a)$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \qquad (2.4.4b)$$

where,

(i) E, F, G, H are real symmetric positive semi-definite matrices of order nxn,

(ii) $C'X - X'EX - (X'GX)^{1/2} + \alpha > 0$, $d'X - X'FX - (X'HX)^{1/2} + \beta > 0$; over the set of feasible solutions (2.4.4b)

(iii) Other symbols having the same meanings as in problem (2.4.1).

In [25] , the problem (2.4.4) was reduced to another problem in which the O.F. is CX but the constraint set is not convex. Here we see that problem (2.4.4) is a particular case of main problem considered in this chapter, and if we assume the matrix E,F,G,H to be positive definite then results established in Section I through III of this chapter hold for problem (2.4.4).

CHAPTER - III

NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING

INTRODUCTION:

An extensive literature is available for solving a L.F.F.P.P. [114,149,150,45,61,81,115,120,122,98,99,164,52 etc.], in which the O.F. is taken to be the ratio of two linear functions and the constraints are taken to be linear. However, a large number of problems of practical interest are not representable by L.F.F.P. model. Therefore, problems more general than L.F.F.P. had to be considered. To include the names for special type of

non-linear fractional programming problems we have Kanti Swarup [121,123,132] , Jagannathan [116,117] and Aggarwal [3,132] . Recently, Dinkelbach [62] , has considered the general N.L.F.P.P. and has given a parametric approach to obtain the solution of such a problem.

The present chapter is concerned with a variety of topics related with certain theoretical aspects of fundamental nature for the problem of N.L.F.P.P. in which the objective function, which is neither convex nor concave, is the ratio of a non-linear function to another non-linear (or linear) function, with their nature, regarding convexity and concavity, properly specified (and if needed assumed to be differentiable) and the constraint set is a convex set (If needed stated to be constrained by properly specified functions). The chapter is divided into five major sections. Sections I through III contain the N.L.F.F.P.P. in which the O.F. is the ratio of a convex function to a concave function (which can be linear also) and Section IV contains the N.L.F.F.P.P. in which the O.F. is the ratio of a concave function to a convex function (which can be linear also). Section V is devoted to particular cases of N.L.F.F.P. Section I mainly contains that, under suitable assumptions for such a problem; (i) Every local minimum is a global minimum; (ii) The set of points, at which the global minimum is taken on, is a convex set; (iii) if the O.F. has a global maximum in the interior of the set of feasible solutions, then the O.F. is constant throughout the set of feasible solutions; (iv) the global maximum will be taken on at one or more of the extreme points of the constraint set. In Section II, the

Kuhn-Tucker conditions [11,142] , for the existence of an optimal solution are shown to be necessary and sufficient. Section III is devoted to certain aspects of duality. In Section IV, under suitable assumptions, results analogous to those established in Sections I through III are stated for the N.L.F.P.P. problem in which the O.F. is the ratio of a concave function to a convex function (which can be linear also). Section V deals with a few particular cases considered by the author and other research workers. As a particular case it has been shown that for obtaining the global optimum of a L.F.N.P.P. a finite iteration technique, similar to 'Simplex Method' can be developed.

SECTION - I

BASIC THEORETICAL RESULTS

In this section we are primarily concerned with the N.L.F.P.P.

Optimize $\quad \varphi(x) = \dfrac{f(x)}{g(x)} \quad$ for $x \in P$, $\qquad$ (3.1.1)

where,

(i) $P \subset S$ is a non-empty closed convex set containing at least two points.

(ii) f is a function which is CX over S and non-negative over P, g is a function which is CV over S and strictly positive P such that the O.F. is neither CX nor CV over P.

(iii) A finite optimum occurs at a finite point of P.

In the present section we shall establish that for the problem (3.1.1); (a) Every local minimum of φ over P is a global minimum; (b) The set of those points in P, at which φ takes on its global

minima, is a convex set; (c) If the O.F. φ has a point of global maximum in the interior of the set P, then the objective function φ is constant over P; (d) The function φ takes on its global maximum at one or more of the extreme points of the set P.

Theorem 1. Every local minimum of φ over P is also a global minimum of φ over P.

Proof. If possible let the theorem be false, such that if $X_0 \in P$ is a local minimum of φ over P and $X_*(\neq X_0) \in P$ is a global minimum of φ over P, we have

$$\varphi(x_0) > \varphi(x_*)$$

i.e.
$$f(x_*) < \varphi(x_0)\, g(x_*) \tag{3.1.2}$$

Consider

$$x_\lambda = \lambda x_* + (1-\lambda)x_0 \quad \in P \quad \forall \lambda \in [0,1] \tag{3.1.3}$$

Therefore, $\forall \lambda \in [0,1]$

$$\varphi(x_\lambda) = \frac{f[\lambda x_* + (1-\lambda)x_0]}{g[\lambda x_* + (1-\lambda)x_0]}$$

Since we have assumed that f is CX and g is CV over S, this implies that f is CX and g is CV on P also (because $P \subset S$). Using, therefore, the convexity and non-negativity of f and concavity and strict positivity of g over P, we obtain $\forall \lambda \in [0,1]$

$$\varphi(x_\lambda) \leq \frac{\lambda f(x_*) + (1-\lambda) f(x_0)}{\lambda g(x_*) + (1-\lambda) g(x_0)}$$

$$= \frac{\lambda f(x_*) + (1-\lambda)\varphi(x_0)\, g(x_0)}{\lambda g(x_*) + (1-\lambda) g(x_0)} \tag{3.1.4}$$

Makin use of (3.1.2) in (3.1.4) we obtain $\forall\ \lambda \in]0,1[$

$$\varphi(x_\lambda) < \frac{\lambda\varphi(x_0)\, g(x_*) + (1-\lambda)\,\varphi(x_0)\, g(x_0)}{\lambda g(x_*) + (1-\lambda)\, g(x_0)}$$

$$= \varphi(x_0)\left[\frac{\lambda g(x_*) + (1-\lambda)\, g(x_0)}{\lambda g(x_*) + (1-\lambda)\, g(x_0)}\right]$$

$$\Longrightarrow \qquad \varphi(x_\lambda) < \varphi(x_0) \tag{3.1.5}$$

But $X_0 \in P$ is a local minimum of φ over P, therefore, there exists on ε-neighbourhood $\mathcal{N}_\varepsilon(X_0)$ of X_0, such that for an appropriate value of λ for $0 < \lambda < 1$ in (3.1.3), it is possible for us to choose X in such a way that $X \in \mathcal{N}_\varepsilon(X_0) \cap P$ and

$$\varphi(x) \geqslant \varphi(x_0) \tag{3.1.6}$$

(3.1.6) shows that (3.1.5) is a contradiction. Thus the result follows.

Theorem 2. The set of those points in P at which φ takes on its global minimum over P, is a convex set.

Proof. Let the set of those points in P at which φ takes on its global minimum be denoted by M. Evidently $M \subset P$.

If M is empty or a singleton, then it is evidently a convex set. If M contains more than one point, then let X_1, X_2 $(X_1 \neq X_2)$ be in $M \subset P$. Then we have

$$\varphi(x_1) = \varphi(x_2) \quad \text{for} \quad x_1, x_2 \in M \subset P. \tag{3.1.7}$$

Now for all $\lambda \in [0,1]$, we have,

$$\varphi[\lambda x_1 + (1-\lambda)x_2] = \frac{f[\lambda x_1 + (1-\lambda)x_2]}{g[\lambda x_1 + (1-\lambda)x_2]} \tag{3.1.8}$$

Exactly as in Theorem 1 above, (3.1.7) and (3.1.8) imply $\forall \lambda \in [0,1]$

$$\varphi[\lambda x_1 + (1-\lambda) x_2] \leq \varphi(x_1) = \varphi(x_2) \tag{3.1.9}$$

But as X_1 and X_2 are the points of global minimum of φ over P, therefore, $\varphi[\lambda x_1 + (1-\lambda) x_2]$, $\forall \lambda \in [0,1]$, cannot be less than $\varphi(x_1) = \varphi(x_2)$. Thus from (3.1.9) we have, $\forall \lambda \in [0,1]$

$$\varphi[\lambda x_1 + (1-\lambda) x_2] = \varphi(x_1) = \varphi(x_2) \Longrightarrow \forall \lambda \in [0,1]$$

$\lambda x_1 + (1-\lambda) x_2 \in M$ i.e. M is a convex set. Hence the result is proved.

We now state a few results in the form of corollaries which are either immediately deducible from results proved above or else can be proved easily.

Corollary 1. If φ takes on its global minimum at two different points in P, then it takes on its global minimum at an infinite number of points in P.

Corollary 2. There cannot be two (or more) points in P at which φ takes on a strong local (and hence a global) minimum.

Corollary 3. If at least one of the functions f and g is a strict function (e.g the function f be a strictly convex function) then the global minimum of φ over P is taken on at a unique point.

Theorem 3. If the function φ has a global maximizing point in the interior of the constraint set P, then φ is constant over P.

Proof. Let us assume that φ is not constant over P, and X_* is an interior point of P such that,

$$\varphi(x_*) = \max_{x \in P} \varphi(x) \tag{3.1.10}$$

Again φ being not constant over P, it is possible for us to find X_1 ($\neq X_*$) $\in$ P such that,

$$\varphi(X_1) < \varphi(X_*) \qquad (3.1.11)$$

Let us choose another point $X_2 \in P$ such that $X_2 \neq X_1$, $X_2 \neq X_*$ and that

$$X_* = \lambda X_1 + (1-\lambda) X_2 \qquad \forall \lambda \in]0,1[\qquad (3.1.12)$$

Now because $X_2 \in P$, therefore, we may have either $\varphi(X_2) = \varphi(X_*)$, or $\varphi(X_2) < \varphi(X_*)$. In the former case we have with the help of (3.1.11)

(1) $\varphi(X_1) < \varphi(X_2)$

Whereas in the latter case we have,

(2) Either (a) $\varphi(X_1) < \varphi(X_2)$

or (b) $\varphi(X_1) > \varphi(X_2)$

or (c) $\varphi(X_1) = \varphi(X_2)$

Thus we can have in all any and only one of the following possibilities:

$$\left.\begin{array}{lll} \text{P-1.} \quad \varphi(X_1) < \varphi(X_2) & ; \text{ when either (i)} & \varphi(X_2) < \varphi(X_*) \\ & \qquad\quad \text{or (ii)} & \varphi(X_2) = \varphi(X_*) \end{array}\right\} \qquad (3.1.13)$$

$$\text{P-2.} \quad \varphi(X_1) > \varphi(X_2) \quad ; \text{ when} \quad \varphi(X_2) < \varphi(X_*) \qquad (3.1.14)$$

$$\text{P-3.} \quad \varphi(X_1) = \varphi(X_2) \quad ; \text{ when} \quad \varphi(X_2) < \varphi(X_*) \qquad (3.1.15)$$

Using (3.1.12) we have $\forall \lambda \in]0,1[$

$$\varphi(X_*) = \frac{f[\lambda X_1 + (1-\lambda) X_2]}{g[\lambda X_1 + (1-\lambda) X_2]} \qquad (3.1.16)$$

Making use of non-negativity and convexity of f and strict positivity and concavity of g in (3.1.16) we have, $\forall\ \lambda \in]0,1[$,

$$\varphi(x_2) \leq \frac{\lambda f(x_1) + (1-\lambda) f(x_2)}{\lambda g(x_1) + (1-\lambda) g(x_2)} \tag{3.1.17}$$

Assuming that possibility P-1 exists, (3.1.17) and (3.1.13) by proceeding exactly as in Theorem 1 above, yield,

$$\varphi(x_*) < \varphi(x_2)$$

which is a contradiction as is evident from (3.1.13).

Similarly, if we assume that either possibility P-2 or possibility P-3, exists, we obtain respectively,

$$\varphi(x_*) < \varphi(x_1) \tag{3.1.18}$$

$$\varphi(x_*) \leq \varphi(x_1) = \varphi(x_2) < \varphi(x_*) \tag{3.1.19}$$

But (3.1.18) and (3.1.19) are also contradictions. Hence the result that φ is constant over P follows.

Theorem 4. If the non-empty convex set P be bounded also, then φ takes on its global minimum over P at one or more of the extreme points of P.

Proof. Using Theorem 3 above and Lemmas 1 and 2 of Chapter II, and arguing exactly on the lines of Theorem 4 in Chapter II, the result follows.

Remark 1. It is very interesting and of great importance to remark over here that in the problem (3.1.1), if in the O.F., the function, g, be a linear function, then keeping the other assumptions theirin unchanged, the condition of negativity upon the function, f, is not necessary. Thus in place of (3.1.1) if we have the following

N.L.F.P.P. . which is stated as,

$$\text{Optimize} \quad \varphi(X) = \frac{f(X)}{g(X)} \qquad \text{for} \quad X \in P, \tag{3.1.20}$$

where,

(i) $P \subset S$ is a non-empty closed convex set containing at least two points.

(ii) f is CX over S, g is linear and strictly positive over P, such that the O.F. φ is neither CX nor CV over P.

(iii) A finite optimum occurs at a finite point of P.

Then the following results can be proved for the problem (3.1.20), exactly on the lines of results proved for problem (3.1.1).

1. Every local minimum of φ is also a global minimum over P. (3.1.21)

2. The set of those points at which φ taken on its global minimum is a convex set. (3.1.22)

3. If φ takes on its global minimum at two different points in P, then it takes on its global minimum at an infinite number of points in P. (3.1.23)

4. There cannot be two or more points in P at which φ takes on a strong local (and hence a global) minimum. (3.1.24)

5. If f be a SX function, then the global minimum of φ is taken on at a unique point. (3.1.25)

6. If the function φ has a global maximising point in the interior of the constraint set P, then φ is constant over P. (3.1.26)

7. If the non-empty convex set P be bounded also, then φ takes on its global minimum over P at one or more of the extreme points of P. (3.1.27)

Remark 2. Although for N.L.F.F.P.P.'s (3.1.1) and (3.1.20), we have established that in each of the problems the O.F. will take on its global maximum at one or more of the extreme points of the set of feasible solutions, yet it is not possible for us to exploit this fact to develop a computational technique, typical of Simplex Method, to find the global maximum of the O.F., since the well known "Adjacent Extreme Point Methods" do not necessarily yield the global maximum and in our case it is possible that φ may take on its local maximum, different from global maximum, at an extreme point of the constraint set. However, to obtain a local maximum of such N.L.F.F.P.P.'s a 'Simplex-Like Technique' can be developed.

SECTION - II

KUHN-TUCKER CONDITIONS FOR OPTIMALITY:

In this section we shall establish the necessary and sufficient conditions, which are similar to those established by Kuhn and Tucker [42], for the existence of the optimal solution of the N.L.F.F.P.P. (3.1.1) with the additional assumption that the set P is given by

$$P = \left\{ X \,;\; g_i(X) \leq 0 \,,\; i \in M = (1, 2, \dots, m) \,;\; X \in S \right\} \tag{3.2.1}$$

where all the functions g_i in the system

$$g_i(X) \leq 0 \quad , \quad i \in M \tag{3.2.2}$$

are QX such that P is a closed convex set. Since the Theorem 1 above (in Section I) gives an interesting property which states that for program (3.1.1) every local minimum is global minimum, therefore, the necessary and sufficient conditions proved below will hold with reference to the global minimum of (3.1.1). We now prove the following theorems.

Theorem 1. If $X_o \in P$ (3.1.1) be a point which is sequentially qualified, [11], for the system (3.2.2) and if all the functions f,g and g_i, $i \in M$; $\in C^1$ at X_o, then the necessary and sufficient conditions for the existence of a minimum to the program given by (3.1.1), (3.2.1) and (3.2.2) are that there exist m scalars λ_i^o, $i \in M$ such that

$$\nabla_x \varphi(x_o) + \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(x_o) = 0 \tag{3.2.3}$$

$$\sum_{i=1}^{m} \lambda_i^o g_i(x_o) = 0 \tag{3.2.4}$$

$$g_i(x_o) \leqslant 0 \qquad i \in M \tag{3.2.5}$$

$$\lambda_i^o \geqslant 0 \qquad i \in M \tag{3.2.6}$$

Proof. To show that the above conditions (3.2.3) through (3.2.6) are necessary, the proof given below follows the lines of [11, Theorem 4] .

Let

$$N_a^o = \left\{ i \ ; \ g_i(x_o) = 0 \right\} \tag{3.2.7}$$

$$N_I^o = \left\{ i \ ; \ g_i(x_o) < 0 \right\} \tag{3.2.8}$$

where $\quad N_a^o = M - N_I^o$

Then, from (2.2.4) through (2.2.8) we obtain that,

$$\lambda_i^o = 0 \qquad \text{for} \quad i \in N_I^o \tag{3.2.9}$$

Now if we assume that the system (3.2.3) (3.2.4) and (3.2.6) has no solution, then we have that the system

$$\nabla_x \varphi(X_0) + \sum_{i \in N_a^0} \lambda_i^0 \nabla_x g_i(X_0) = 0 \tag{3.2.10}$$

$$\lambda_i^0 \geq 0 \qquad i \in N_a^0 \tag{3.2.11}$$

have no solution. Using the Transposition Theorem and the fact that (3.2.10) and (3.2.11) have no solution, it follows that there exists some $Y \in R^n$ such that

$$[\nabla_x \varphi(X_0)]' Y < 0 \tag{3.2.12}$$

$$[\nabla_x g_i(X_0)]' Y \leq 0 \qquad i \in N_a^0 \tag{3.2.13}$$

From (3.2.13) we infer that Y belongs to linearizing cone to (3.2.2) at $X_0 \in P$. Since X_0 is sequentially qualified, this implies that Y belongs to the cone of tangents to P at X_0. Therefore, there exists some sequence (X_p) contained in P and converging to X_0, and some sequence (λ_p) of non-negative numbers, such that the sequence $((X_p - X_0)\lambda_p)$ converges to Y. Again we have,

$$\varphi(X_p) = \varphi(X_0) + [\nabla_x \varphi(X_0)]'(X_p - X_0) + \|X_p - X_0\| \varepsilon \tag{3.2.14}$$

where ε is a scalar, depending on p, and tending to zero as p tends to infinity, and, $\| \quad \|$, denotes the norm. From (3.2.14), we have

$$[\varphi(X_p) - \varphi(X_0)]\lambda_p = [\nabla_x \varphi(X_0)]'(X_p - X_0)\lambda_p + \|(X_p - X_0)\lambda_p\| \varepsilon \tag{3.2.15}$$

Let p become sufficiently large such that $\varepsilon \to 0$ and $(X_p - X_0)\lambda_p$ converges to Y, then $\|(X_p - X_0)\lambda_p\| \varepsilon \longrightarrow \|Y\| \varepsilon$ with $\varepsilon \to 0$,

which implies that the right hand side in (3.2.15) has the sign of $[\nabla_x \varphi(X_0)]' Y$, which by (3.2.12) is negative. This yields that

$$\varphi(X_p) < \varphi(X_0) \tag{3.2.16}$$

since $X_p \in P$, therefore, (3.2.16) is a contradiction to the fact that X_0 is a minimum to the given program. Hence we have the result.

To prove that the conditions (3.2.3) through (3.2.6) are sufficient also we assume that there exist X_0 and λ_i^0, $i \in M$, satisfying the conditions (3.2.3) through (3.2.6). We have now,

$$\nabla_x \varphi(X_0) = \frac{g(X_0)\,\nabla_x f(X_0) - f(X_0)\,\nabla_x g(X_0)}{[g(X_0)]^2}$$

$$\Rightarrow (X - X_0)' \nabla_x \varphi(X_0) = \frac{g(X_0)(X-X_0)'\nabla_x f(X_0) - f(X_0)(X-X_0)'\nabla_x g(X_0)}{[g(X_0)]^2} \quad \text{for } X \in P. \tag{3.2.17}$$

Using the non-negativity and convexity of f and strict positivity and concavity of g we obtain from (3.2.17),

$$(X - X_0)' \nabla_x \varphi(X_0) \leqslant \frac{g(X_0)\,[f(X) - f(X_0)] - f(X_0)\,[g(X) - g(X_0)]}{[g(X_0)]^2}$$

$$= \frac{g(X)}{g(X_0)} \left[\varphi(X) - \varphi(X_0)\right] \tag{3.2.18}$$

From (3.2.18) we get, for $X \in P$,

$$\varphi(X) - \varphi(X_0) \geqslant \frac{g(X_0)}{g(X)} \left[(X - X_0)' \nabla_x \varphi(X_0)\right] \tag{3.2.19}$$

From (3.2.3) we have

$$(X-X_o)'\nabla_x \varphi(X_o) + \sum_{i=1}^{m} \lambda_i^o (X-X_o)'\nabla_x g_i(X_o) = 0 \qquad (3.2.20)$$

Also we already have from (3.2.9) that

$$\lambda_i^o = 0 \qquad \text{for } i \in N_I^o \qquad (3.2.21)$$

Using (3.2.1) and (3.2.7) for $X \in P$, we have

$$g_i(X) \leq g_i(X_o) \qquad i \in N_a^o \qquad (3.2.22)$$

Quasi-convexity assumption on functions g_i's, $i \in M$, yields

$$g_i(X) \leq g_i(X_o) \Rightarrow (X-X_o)'\nabla_x g_i(X_o) \leq 0 \qquad (3.2.23)$$

Thus (3.2.22) and (3.2.23) yield,

$$(X-X_o)'\nabla_x g_i(X_o) \leq 0 \qquad i \in N_a^o \text{ and } X \in P. \qquad (3.2.24)$$

Making use of (3.2.6) and (3.2.24) we have for $X \in P$

$$\sum_{i \in N_a^o} \lambda_i^o (X-X_o)'\nabla_x g_i(X_o) \leq 0 \qquad (3.2.25)$$

From (3.2.20) we evidently have for $X \in P$

$$\sum_{i \in N_I^o} \lambda_i^o (X-X_o)'\nabla_x g_i(X_o) = 0 \qquad (3.2.26)$$

Combining (3.2.25) and (3.2.26) we obtain

$$\sum_{i=1}^{m} \lambda_i^o (X-X_o)'\nabla_x g_i(X_o) \leq 0 \qquad \text{for } X \in P. \qquad (3.2.27)$$

(3.2.19) and (3.2.20) in conjunction with (3.2.27) yield

$$\varphi(X) \geq \varphi(X_o) \qquad \text{for } X \in P. \qquad (3.2.28)$$

i.e. $X_0 \in P$ is a global minimum of φ over P. Hence the result.

Remark 1. In above either N_I° or N_a° can happen to be a null set. To modify the proof, in the former case we omit (3.2.21), (3.2.26) and the a ssociated references and in the latter case we omit (3.2.22), (3.2.24) and (3.2.25).

Corollary 1. If for $X_0 \in P$, $\nabla_x \varphi(X_0) = 0$, then φ takes on its global minimum over P, at X_0.

Proof. We have $\nabla_x \varphi(X_0) = 0$ for $X_0 \in P$.

Therefore, for all $X \in P$; $(X - X_0)' \nabla_x \varphi(X_0) = 0$ where $X_0 \in P$. (3.2.28)

Using (3.2.28) in (3.2.19) we obtain $\varphi(X) \geq \varphi(X_0)$ for all $X \in P$ i.e. $X_0 \in P$ is a global minimum of φ over P.

Wolfe [183] has proved that if φ be a concave function on a convex set S, then any point having one of the three properties of being a stationary point, a local solution, or a global solution, also has the other two properties. Below we prove the theorem for a more general function given in (3.2.1).

Theorem 2. For the function $\varphi = f/g$ given by (3.2.1) and satisfying the assumptions stated in Section I and II, any point in P having one of the three properties of being a stationary point, a local solution, or a global solution also has the other two properties.

Proof. The result follows easily on using the Theorem 1 of Section I, Theorem 1 and its Corollary 1 of Section II.

Remark 2. As in Section I, Remark 1, if we consider the N.L.F.F.P.P. to be given by (3.1.20), then also we can similarly prove Theorem 1 and 2 and other associated results, under relavent assumption stated earlier.

SECTION III

SOME ASPECTS OF DUALITY:

In this section we shall investigate certain duality aspects, similar to those investigated for N.L.F.P.P., for the N.L.F.P.P. considered in Section I and II of this chapter. Mathematical formulations of the (PP) and (DP) for N.L.F.P.P. considered above is as follows.

PRIMAL PROBLEM (PP)

$$\text{Minimize} \quad \varphi(X) = \frac{f(X)}{g(X)} \quad \text{for} \quad X \in P, \tag{3.3.1}$$

where,

(i) the set $P \subset S \subset R^n$ is defined as follows:

$$P = \left\{ X \; ; \; g_i(X) \leq 0, \; i \in M = (1, 2, ----, m) \; ; \; X \in S \right\} \tag{3.3.2}$$

it being assumed that the functions g_i for all $i \in M$ are QX such that the set P is a closed convex set.

(ii) X_o P is the optimal solution of (3.3.1) and is sequentially qualified for the system $g_i(X) \leqslant 0$, $i \in M$.

(iii) functions g_i for all $i \in M$ and $f, g \in C^2$ at X_o.

(iv) f is convex over S and non-negative over P, g is strictly positive over P and convex over S, such that the O.F. φ is neither CX nor CV over P.

DUAL PROBLEM (DP):

For the problem (2.3.1), according to Wolfe 181 the (DP) is:

$$\text{Maximize} \; F(X, \lambda) = \varphi(X) + \sum_{i=1}^{m} \lambda_i g_i(X) \quad \text{for} \; (X, \lambda) \in D \tag{3.3.3}$$

where,

(i) the set $D \subset R^{n+m}$ is given by,

$$D = \left\{(X, \lambda) \; ; \; \nabla_x \varphi(X) + \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X) = 0 , \lambda \geqslant 0 \right\} \qquad (3.3.4)$$

(ii) $\lambda = (\lambda_1, \lambda_2, ----, \lambda_m)'$ is an mx1 column vector $\in R^m$.

(iii) $(X_0, \lambda_0) \in D$ is sequentially qualified for the system

$$\nabla_x \varphi(X) + \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X) = 0,$$

We now prove the following two theorems which connect (PP) and (DP). Theorem 2, called, "The Converse Duality Theorem", is a generalisation of theorems of Mangasarian [147] and Huard [109] for convex programming to more general case of N.L.F.F.P.

Theorem 1. If X_0 minimizes (PP), then there exists a point $(X_0, \lambda_0) \in D$ such that $F(X_0, \lambda_0) = \min\limits_{X \in P} \varphi(X)$ and $\max\limits_{\substack{(X_0, \lambda) \in D \\ X_0 \in P}} F(X_0, \lambda) \leq F(X_0, \lambda_0)$ for all $(X_0, \lambda) \in D$.

Proof. Since X_0 is the minimizing point of (PP), therefore, by making use of Theorem 1 of Section II, we have that it is necessary and sufficient that there exists a vector $\lambda_0 = (\lambda_1^0, \lambda_2^0, ----, \lambda_m^0)' \in R^m$ such that

$$\nabla_x \varphi(X_0) + \sum_{i=1}^{m} \lambda_i^0 \nabla_x g_i(X_0) = 0 \qquad (3.3.5)$$

$$\sum_{i=1}^{m} \lambda_i^0 \, g_i(X_0) = 0 \qquad (3.3.6)$$

$$g_i(X_0) \leqslant 0 \qquad i \in M \qquad (3.3.7)$$

$$\lambda_i^0 \geqslant 0 \qquad i \in M \qquad (3.3.8)$$

Because it is assumed that (X_o, λ_o) is sequentially qualified for the system $\nabla_x \varphi(X) + \sum_{i=1}^{m} \lambda_i \nabla_x g_i(X) = 0$, therefore, the necessary conditions for (X_o, λ_o) to be a solution of the (DP) are, as given by [11, 142], that there must exist a vector $V_o \in R^n$, such that,

$$\nabla_x \varphi(X_o) + \nabla_x \lambda_o' G(X_o) + \nabla_x V_o' \nabla_x \varphi(X_o) + \nabla_x V_o' \nabla_x \lambda_o' G(X_o) = 0 \qquad (3.2.12)$$

$$G(X_o) + \nabla_\lambda V_o' \nabla_x \lambda_o' G(X_o) \leq 0 \qquad (3.3.13)$$

$$\nabla_x \varphi(X_o) + \nabla_x \lambda_o' G(X_o) = 0 \qquad (3.3.14)$$

$$\lambda_o' G(X_o) + \lambda_o' \nabla_\lambda V_o' \nabla_x \lambda_o' G(X_o) = 0 \qquad (3.3.15)$$

$$\lambda_o \geq 0 \qquad (3.3.16)$$

Making use of (3.4.12) and (3.3.14) we get

$$\nabla_x V_o' \nabla_x \varphi(X_o) + \nabla_x V_o' \nabla_x \lambda_o' G(X_o) = 0$$

$$\Longrightarrow \quad \nabla_x V_o' \nabla_x F(X_o, \lambda_o) = 0 \qquad (3.3.17)$$

Since we have assumed that the Hessian of $F(X, \lambda_o)$ at X_o is non-singular, therefore, by using (3.3.17) and Lemma 1 of Section III Chapter II, we obtain

$$V_o = 0 \qquad (3.3.18)$$

Combining (3.3.12) and (3.3.18) we obtain,

$$\nabla_x \varphi(X_o) + \sum_{i=1}^{m} \lambda_i^o \nabla_x g_i(X_o) = 0 \qquad (3.3.19)$$

(3.3.15) in conjunction with (3.3.17) yields,

$$\sum_{i=1}^{m} \lambda_i^o g_i(X_o) = 0 \qquad (3.3.20)$$

From (3.3.18) and (3.3.13) we have,

$$g_i(X) \leqslant 0 \qquad i \in M \tag{3.3.21}$$

Comparing the conditions (3.3.19) through (3.3.21) and (3.3.16) with the conditions (3.2.3) through (3.2.6) we conclude that X_o P is a minimum of (PP). Hence the result.

Remark 1. We can similarly, under relevant assumptions, prove the above results, of the present section, for the problem given by (3.1.20) in which in the objective function, there is no non-negativity restriction on the CX function f, however, the function g which is strictly positive is assumed to be linear.

SECTION IV

ANALOGOUS NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING:

In this section our main concern is with the N.L.F.F.P.P.

$$\text{Optimise} \quad \varphi(X) = \frac{f(X)}{g(X)} \qquad \text{for } X \in P, \tag{3.4.1}$$

where,

(i) $P \subset S$ is a non-empty closed convex set containing at least two points and if necessary assumed to be given by

$$P = \left\{ X \; ; \; g_i(X) \leqslant 0 \, , \; i \in M \; ; \; X \in S \right\} \tag{3.4.2}$$

where all the functions g_i in the system

$$g_i(X) \leqslant 0 \qquad i \in M \tag{3.4.3}$$

are QX.

(ii) f is CV over S and non-negative over P, g is CX over S and strictly positive over P; and if g is linear and strictly positive then there is no non-negativity restriction on f.

(iii) A finite optimum occurs at a finite point of P.

For the above problem we now state the following results which can be proved similarly as in the earlier sections of this chapter.

1. Every local maximum of φ over P is a global maximum also (3.4.4)

2. The set of those points in P, at which φ takes on its global maxima, is a convex set. (3.4.5)

3. If φ takes on its global maximum at two different points in P, then it takes on its global maximum at an infinite number of points. (3.4.6)

4. There cannot be two (or more) points in P at which φ takes on a strong local (and hence a global) maximum. (3.4.7)

5. If at least one of the functions f and g is a strict function (e.g. the function f is strictly concave), then the global maximum of over P is taken on at a unique point. (3.4.8)

6. If the function φ has a global minimizing point in the interior of the constraint set P, then φ is constant over P. (3.4.9)

7. If the set P be bounded also, then φ takes on its global minimum over P at one or more of the extreme points of P. (3.4.10)

8. If $X_0 \in P$ (3.4.2) be a point which is sequentially qualified for the system (3.4.3) and if all the functions f, g and $g_i, i \in M; \in C^1$ at X_0, then the necessary and sufficient conditions for the existence of a maximum to the program (3.4.1) are that there exist m scalars $\lambda_i, i \in M$ such that,

$$\left.\begin{array}{ll} \nabla\varphi(x_0) - \sum_{i=1}^{m} \lambda_i^0 \nabla_x g_i(x_0) = 0 & \\ \sum_{i=1}^{m} \lambda_i^0 g_i(x_0) = 0 & \\ g_i(x_0) \leq 0 & i \in M \\ \lambda_i^0 \geq 0 & i \in M \end{array}\right\} \qquad (3.4.11)$$

DUALITY AS NOTS:

For the problem (3.4.1) we have the primal and dual problems as follows:

(PP) Maximize $\varphi(x) = \dfrac{f(x)}{g(x)}$ for $x \in P$, (3.4.12)

where in addition to the assumption stated for problem (3.4.1) it is further assumed that (i) $X_o \in P$ is the optimal solution of (3.4.12) and is sequentially qualified for the system (3.4.3), (ii) functions g_i for all $i \in M$ and $f, g; \in C^2$ at X_o.

(DP) Minimize $F(x,\lambda) = \varphi(x) - \sum_{i=1}^{m} \lambda_i g_i(x)$ for $(x,\lambda) \in D$, (3.4.13)

where,

(i) the set $D \subset R^{n+m}$ is given by,

$$D = \left\{(x,\lambda)\ ;\ \nabla_x \varphi(x) - \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x) = 0\ ,\ \lambda \geq 0 \right\} \qquad (3.4.14)$$

(ii) $(X_o, \lambda_o) \in D$ is sequentially qualified for the system

$$\nabla_x \varphi(x) - \sum_{i=1}^{m} \lambda_i \nabla_x g_i(x) = 0$$

For problems (3.4.12) and (3.4.13) we have now following two theorems analogous to Theorems 1 and 2 of Section III.

Theorem 1. If X_o maximizes (PP), then there exists a point $(X_o, \lambda_o) \in D$ such that $F(X_o, \lambda_o) = \underset{X \in P}{Max}\, \varphi(X)$ and $\underset{\substack{(X_o,\lambda)\in D \\ X_o \in P}}{Min} F(X_o, \lambda) \geq F(X_o, \lambda_o)$ for all $(X_o, \lambda) \in D$.

Theorem 2 **The Converse Duality Theorem** : If (X_o, λ_o) is an optimal solution to (DP) and $F(X, \lambda_o)$ is twice continuously differentiable with respect to X in a neighbourhood of X_o, and if the Hessian of $F(X, \lambda_o)$ with respect to X is non-singular at X_o, then X_o optimizes (PP).

SECTION - V

PARTICULAR CASES: Below we now consider a few particular cases of the fractional functional programming problem considered by other research workers and the author.

1. Aggarwal [1] , considered the following N.L.F.F.P.P.

Minimize
$$\varphi(X) = \frac{X'HX + C'X + \alpha}{d'X + \beta} \tag{3.5.1a}$$

subject to

$$\left.\begin{aligned} AX &\geq b \\ X &\geq 0 \end{aligned}\right\} \tag{3.5.1b}$$

where,

(i) c,d are nx1 column vectors, b is an mx1 column vector.

(ii) H is a real symmetric positive semi-definite matrix of order nxn.

(iii)A is an mxn order matrix; α, β are arbitrary scalar constants.

(iv) $d'X + \beta > 0$ over the constraint set.

and proved that the O.F. φ , in (3.5.1a), is FQV. Here we see that the problem is a particular case of problem (3.1.20).

2. Aggarwal [3] considered the problem

$$\text{Maximize} \quad \varphi(X) = \frac{C'X - (X'HX)^{1/2} + \alpha}{d'X + \beta} \tag{3.5.2a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{3.5.2b}$$

where all the symbols and assumptions are as in problem (3.5.1). Aggarwal, under the further assumption that the set of feasible solutions is regular, showed that the solution of the problem (3.5.2) can be obtained by another convex programming problem. But here it is seen that the above problem is a particular case of the problem (3.4.1).

3. Bector [22] , considered the following N.L.F.P.P.'s which are more general than (3.5.1) and (3.5.2) but are particular cases of the general problems considered in the present chapter.

(i) Minimize $$\varphi(X) = \frac{X'HX + (X'GX)^{1/2} + C'X + \alpha}{d'X + \beta} \tag{3.5.3a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{3.5.3b}$$

(ii) Maximize $$\varphi(X) = \frac{X'EX - (X'HX)^{1/2} + C'X + \alpha}{d'X + \beta} \tag{3.5.4a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{3.5.4b}$$

where the matrices G and E are respectively positive semi-definite and negative semi-definite each of order nxn and other symbols and assumptions are as in problem (3.5.1), (3.5.2).

Evidently problem (3.5.3) is a particular case of problem (3.1.2) and problem (3.5.4) is a particular case of (3.4.1). However, under the assumption that the constraint set is regular, the above problems (3.5.3) and (3.5.4) can also be solved by reducing them to convex programming problems.

4. Here we consider a slightly more generalised form of problems (3.5.3) and (3.5.4) but which cannot be reduced to convex programming problems.

(i) Minimize $\varphi(X) = \dfrac{X'HX + (X'GX)^{1/2} + c'X + \alpha}{X'EX - (X'FX)^{1/2} + d'X + \beta}$ (3.5.5a)

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \qquad (3.5.5b)$$

(ii) Maximize $\varphi(X) = \dfrac{X'EX - (X'FX)^{1/2} + c'X + \alpha}{X'GX + (X'HX)^{1/2} + d'X + \beta}$ (3.5.6a)

subject to

$$\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned} \qquad (3.5.6b)$$

where, the matrix F is a positive semi-definite matrix; in problem (3.5.5), $X'HX + (X'GX)^{1/2} + c'X + \alpha \geq 0$, $X'EX - (X'FX)^{1/2} + d'X + \beta > 0$ and in problem (3.5.6) $X'EX - (X'FX)^{1/2} + c'X + \alpha \geq 0$, $X'GX + (X'HX)^{1/2} + d'X + \beta > 0$ other symbols and assumptions are same as in problems (3.5.3), (3.5.4).

We see that problems (3.5.5) and (3.5.6) are particular cases of problems (3.1.1) and (4.1.1) respectively.

Remark 1. In problem (3.5.5) if we take matrices G and F as null matrices and in problem (3.5.6), the matrices F and H as null matrices, we obtain the N.L.F.F.P.P. of the type considered by Kanti Swarup [121] .

5. We now consider the well known problem of L.F.F.P. and show that most of its very important and interesting results follow as particular cases to more general theorems proved in the present chapter. A L.F.F.P.P. is stated as follows:

$$\text{Optimize} \quad \varphi(X) = \frac{c'X + \alpha}{d'X + \beta} \tag{3.5.7a}$$

subject to

$$\left.\begin{aligned} AX &\leq b \\ X &\geq 0 \end{aligned}\right\} \tag{3.5.7b}$$

where,

(i) Symbols have the same meanings as already explained.

(ii) A finite maximum occurs at a finite point of the set of feasible solutions (3.5.7b) which is assumed to be bounded and non-empty.

(iii) $d'X + \beta > 0$, throughout the set of feasible solutions.

If we set $f(X) = c'X + \alpha$, then we see that in (3.5.7a) $f(X)$ is both a CX as well as a CV function. Thus the O.F. in problem (3.5.7) is such that it is the ratio of a, CX (CV), function to a strictly positive linear function. Therefore, here as a particular case of problem (3.1.20) and analogous problem in Section IV we have in addition to other results deducible from general cases, the following most interesting properties of L.F.F.P.P.

(i) Every local minimum (maximum) of φ over (3.5.7b) is also a global minimum (maximum).

(ii) The set of those points at which φ takes on its global minimum (maximum) is a convex set and, therefore, if φ takes on its global minimum (maximum) at two different points, then it takes on its global minimum (maximum) at an infinite number of points.

(iii) If φ takes on its global minimum (maximum) at an interior point of the constraint set, then it is constant throughout the constraint set.

(iv) The O.F. φ takes on its global minimum (maximum) at one or more of the extreme points of the constraint set (3.5.7b).

A slightly more general form of (3.5.7) but yet a very special case of the N.L.F.F.P.P. considered in this chapter is considered by Kanti Swarup [123,130]

$$\text{Maximise} \quad \varphi(X) = \frac{c'X + \alpha}{d'X + \beta} \tag{3.5.8a}$$

subject to

$$g_i(X) \leq 0 \qquad ; \; i \in M \tag{3.5.8b}$$

where in [123, 130] it is assumed that all g_i for $i \in M$ is are convex functions. It is easily seen that the results deduced by Kanti Swarup in [123,130] follow as particular cases of the problem of N.L.F.F.P. considered in the present chapter.

CHAPTER - IV

SOME PROPERTIES OF EXPLICIT QUASI-CONVEX (QUASI-CONVEX) AND STRONG PSEUDO-CONVEX FUNCTIONS; NATURE OF, PRODUCTS, QUOTIENTS, RATIONAL POWERS AND COMOSITION OF CONVEX-LIKE FUNCTIONS; CHARACTERISATION OF SOME PROGRAMMING PROBLEMS; AND CONVEX FRACTIONAL FUNCTIONAL PROGRAMMING

INTRODUCTION:

Recently much attention has been paid to the class of programming problems which are more general than convex programming. Arrow and Enthoven [8] , and Arrow Hurwicz and Uzawa [7] , discussed certain properties of QV and QX functions with special reference to applications to M.P.P.'s, and Economics. Mangasarian [148] introduced the idea of PCX and PCV functions and investigated some of their properties in the context of M.P.P.'s. Bela Martos [151,152*] investigated the necessary

* The author received the reprint of this paper from Prof. Bela Martos, in July, 1968, when the work contained in this Thesis was already finished.

and sufficient conditions in terms of QX (QV) and EQV(EQX) functions under which a M.P.P. with linear constraints could be solved with the help of a technique similar to 'Simplex Method'.

The main purpose of this chapter is to establish certain properties of fundamental importance of EQX(EQV) and SPCX(SPCV) functions and characterize Indefinite Functional Programs and Fractional Functional Programs considered in earlier chapters and some more general M.P.P.'s, as Explicit Quasi-Concave Programming Problems (E.V.P.P.) or Explicit Quasi-Convex Programming Problems (E X.P.P.) and Pseudo-Concave Programming Problems (P V P.P.) or Pseudo-Convex Programming Problems (P X P.P.), by making a systematic study of the nature of products, quotients, rational powers of CXL functions and composite functions. A few new concepts of Strong Pseudo-Convex (SPCX), Weakly Convex-Like (WCXL), Weakly Strong Pseudo-Convex-Like (WSPCXL) and Quasi-Convex-Like (QXL) real valued, differentiable, scalar functions have been introduced and their relations among themselves and with already existing classes of CXL functions have been established. SPCX functions have been defined on a convex set whereas for defining other functions, the underlying domain need not be convex. It is shown that the concept strong pseudo-convexity is stronger than pseudo-convexity but weaker than convexity. Also an attempt is made to characterize a special class of fractional functional programs as Convex Programs. Although the theory developed in this chapter is not directly concerned with the computational techniques, yet it is of fundamental importance in developing in the next chapter, a numerical method typical of

'Method of Feasible Directions [184,185] ', for a class of non-convex programming problems and a finite iteration technique similar to that of Beale [15] , for obtaining the global optimum for another type of non-convex programming problem.

The chapter is divided into five sections. In Section I certain theorems of fundamental importance regarding EQX(EQV), QX(QV) and PCX(PCV) functions are proved. Section II is devoted to investigate certain properties of newly defined functions named as SPCX, WCXL, WSPCXL and QXL functions. Section III deals with a systematic study of the nature of products, quotients, rational powers of CXL functions and composite functions. In Section IV, problems of N.L.I.F.P., N.L.F.F.P. studied in previous chapters and some more general M.P.P.'s have been characterized as EV.P.P. (EX.P.P.) and PV.P.P. (PX.P.P.). N.L.F.F.P.P. has also been characterized as ' Strong Pseudo Convex Programming Problem' and L.F.F.P.P. is shown to a problem of Strong Pseudo-Monotonic Programming. In Section V, an attempt is made to bring certain problems of fractional functional programming and indefinite functional programming within the framework of convex programming. The most important and interesting result proved in this section is that the ratio of the square of a non-negative CX function to a strictly positive CV function (if the convex and concave functions be linear then the condition of non-negativity on the CX function may be omitted) is a CX function. This result helps in reducing certain non-convex programs, considered in this section, to convex programs.

Note. Throughout this chapter S will denote a closed convex subset of R^n, having at least two points.

SECTION - I

Bela Martos [151], among other results proved that, "If a function φ which is continuous on a non-empty compact convex polyhedral set $L \subset R^n$ be EQX on L, then every local minimum of over L is global minimum also. This result can be improved as in Theorem 1 below.

Theorem 1. If a continuous function φ be EQX (EQV) on S then every local minimum (maximum) of φ over S is a global minimum (maximum) also.

Proof. Let us assume that φ is EQX on S. If possible, let the assertion of the theorem be false, such that if $X_o \in S$ is a local minimum of φ and $X_* (\neq X_o) \in S$ is is a global minimum of φ over S, we have

$$\varphi(X_o) > \varphi(X_*) \tag{4.1.1}$$

Consider $\qquad X_\lambda = \lambda X_* + (1-\lambda) X_o \qquad$ for all $\lambda \in]0,1[$. (4.1.2)

Since S is convex, therefore for all $\lambda \in]0,1[$, $X \in S$.

Using the explicit quasi-convexity of φ over S, we get

$$\varphi(X_\lambda) < max[\varphi(X_o), \varphi(X_*)] \tag{4.1.3}$$

Now, (4.1.1) and (4.1.3) yield

$$\varphi(X_\lambda) < \varphi(X_o) \tag{4.1.4}$$

But $X_o \in S$ is assumed to be a local minimum φ over S, and φ is

continuous over S, therefore there exists an ε-neighbourhood $\mathcal{N}_\varepsilon(X_0)$ of X_0, such that for an appropriate value of $\lambda \in \,]0,1[$ in (4.1.2) it is always possible for us to choose X_λ in such a way that $X_\lambda \in \mathcal{N}_\varepsilon(X_0) \cap S$ and

$$\varphi(X_\lambda) \geqslant \varphi(X_0) \tag{4.1.5}$$

From (4.1.5) and (4.1.4) we see that (4.1.4) is a contradiction. Thus the result follows.

Similarly when φ is EQV, it can be proved that every local maximum of φ over S is global maximum also.

For a convex programming problem (1.2.17) it is stated in (1.2.20) that the set of those points in a convex set, at which the convex O.F. takes on its global minima, is a convex set. This result can be improved as in Theorem 2 below.

Theorem 2. If a continuous function φ be QX (QV) on S, then set of those of S at which φ takes on its global minimum (maximum) over S, is a convex set.

Proof. Let us assume that φ is QX, and let M denote the set of those points at which φ takes on its global minimum. If M is empty or a singleton, then M is trivially a convex set. So, let us assume now that X_1, X_2, $X_1 \neq X_2$ are in M, so that

$$\varphi(X_1) = \varphi(X_2) \tag{4.1.6}$$

Since φ is QX on S, therefore, for all $\lambda \in [0,1]$, (4.1.6) implies that

$$\varphi\left[\lambda X_1 + (1-\lambda) X_2\right] \leqslant \varphi(X_1) = \varphi(X_2) \tag{4.1.7}$$

Since X_1, X_2 are points of global minima, therefore, in (4.1.7)

$\varphi[\lambda X_1 + (1-\lambda)X_2] < \varphi(X_1) = \varphi(X_2)$ is not possible, Hence for all $\lambda \in [0,1]$, we have

$$\varphi[\lambda X_1 + (1-\lambda)X_2] = \varphi(X_1) = \varphi(X_2)$$

This implies that $\lambda X_1 + (1-\lambda)X_2 \in M$ for all $\lambda \in [0,1]$.

Hence M is a convex set.

The following two theorems extend the results (1.2.21) and (1.2.22) of 'Convex Programming' to a programming problem in which the objective function may be EQX(EQV).

Theorem 3. If a continuous function φ be EQX(EQV) over the set S, and if it attains its global maximum (minimum) at an interior point of S, then it is constant in S.

Proof. Let us assume that φ is EQX and is not constant in S. Let X_* be an interior point of S such that,

$$\varphi(X) = \max_{x \in S} \varphi(X) \tag{4.1.8}$$

The assumption that φ is not constant in S implies that it is possible for us to find a point X_1 (different from X_*) in S such that

$$\varphi(X_1) < \varphi(X_*) \tag{4.1.9}$$

Let X_2 be another point in S such that $X_2 \neq X_1$, $X_2 \neq X_*$ and $\varphi(X_1) \neq \varphi(X_2)$, and

$$X_* = \lambda X_1 + (1-\lambda)X_2 \qquad \text{for all} \quad \lambda \in \,]0,1[\tag{4.1.10}$$

Since $X_2 \in S$, therefore, either $\varphi(X_2) = \varphi(X_*)$ or $\varphi(X_2) < \varphi(X_*)$.

In the former case we have on making use of (4.1.9)

(1). $\varphi(X_1) < \varphi(X_2)$

whereas in the later case we have

(2). either (a) $\varphi(X_1) < \varphi(X_2)$

or (b) $\varphi(X_1) > \varphi(X_2)$

Thus in all we can have any one of the following two mutually exclusive possibilities.

$$\left.\begin{array}{ll} \text{P-1.} \quad \varphi(X_1) < \varphi(X_2); & \text{when either (i) } \varphi(x_2) < \varphi(x_*) \\ & \text{or (ii) } \varphi(x_2) = \varphi(x_*) \end{array}\right\} \quad (4.1.11)$$

$$\text{P-2.} \quad \varphi(X_1) > \varphi(X_2); \quad \text{when} \quad \varphi(x_2) < \varphi(x_*) \qquad (4.1.12)$$

Assuming that P-1 exists and making use of the fact that φ is EQX over S, we obtain,

$$\varphi(x_*) < \varphi(x_2)$$

which is a contradiction to (4.1.11)

Similarly when we assume that the possibility P-2 exists, explicit quasi-convexity of φ over S gives

$$\varphi(x_*) < \varphi(x_1)$$

which again is a contradiction to (4.1.9). Hence the result follows. Similarly the result is proved when we assume that φ is EQV over S.

Theorem 4. If the set S be bounded also and if a continous function φ be EQX(EQV) over S, then its global maximum (minimum) will be taken on at one or more extreme points of S.

Proof. Let φ be EQX(EQV) over S. Using Theorem 3 above and Lemmas 1 and 2 of Chapter 2, the result follows, following exactly the lines of proof of Theorem 4 in Chapter II, Section I.

Remark 1. Although, in this section, we have established that when φ is EQX(EQV) over a non-empty compact convex set S, then it will take on its global maximum (minimum) at one or more extreme points of S, yet it is not possible to exploit this fact to develop a computational technique typical of 'Simplex Method' to find the global maximum (minimum) of φ, since the well known "Adjacent Extreme Point Methods" do not necessarily yield the global maximum (minimum) and in our case it is possible that φ may take on its local maximum (minimum), different from global maximum (minimum), at an extreme point of the constraint set. However, to obtain a local maximum (minimum) of φ over S, a 'Simplex-Like Technique' can be developed. In case the function φ be both EQX and EQV over S, then it is possible to obtain the global optimum of φ over S, with the help of a 'Simplex-Like Procedure'.

We now consider the following M.P.P.,

$$\text{Minimise} \quad \varphi(X) \quad \text{for} \quad X \in P, \tag{4.1.13}$$

where $P \subset S$ is the set of feasible solutions given by

$$P = \left\{X ; g_i(X) \leq 0,\ i \in M = (1, 2, \dots, m),\ X \geq 0,\ X \in S\right\} \tag{4.1.14}$$

Kunzi, Krell & Oettli [140], assuming the functions φ and g_i, $i \in M$ to be convex consider the following simplified programming problem,

$$\text{Minimise} \quad \varphi(X) \quad \text{for} \quad X \in P_a, \tag{4.1.15}$$

where, $$P_a = \{X \; ; \; g_i(X) \leq 0, \; i \in N_a, \; x_j \geq 0, \; j \in J_a\}$$

$$N_a = \{i \; ; \; g_i(X) = 0\}, \quad N_I = \{i \; ; \; g_i(X) < 0\}$$

$$N_a = M - N_I$$

$$J_a = \{j \; ; \; x_j = 0\}, \quad J_I = \{j \; ; \; x_j > 0\}$$

$$J = (1, 2, ----, n), \quad J_a = J - J_I$$

and proved that if X_* is an optimal solution of (4.1.13) then it is also an optimal solution of problem (4.1.15). Here we show that the result still holds if we assume that the function φ is EQX, and all the functions g_i, $i \in M$ are QX.

Proof. Assume that X_* is not an optimal solution of (4.1.15); then there exists a point X with $x_j \geq 0$, $j \in J_a$, $g_i(X) \leq 0$ for $i \in N_a$ and $\varphi(X) < \varphi(X_*)$. Then for sufficiently small $\lambda > 0$, we have

$$g[\lambda X + (1-\lambda) X_*] \leq 0 \qquad \text{for all } i \in M.$$

$$\lambda x_j + (1-\lambda) x_j^* \geq 0 \qquad \text{for all } j \in J$$

and because of the explicit-quasi convexity of φ,

$$\varphi[\lambda X + (1-\lambda) X_*] < \varphi(X_*)$$

which is a contradiction to the fact that X_* is a global minimum of (4.1.13). Hence the result follows.

Remark 2. The converse of the above result is obviously not true. A solution of (4.1.15) can violate some of the dropped constraints of (4.1.13).

SECTION - II

The purpose of this section is to introduce 'Strong Pseudo-Convex' functions, 'Weakly Convex-Like' functions, 'Weakly Strong Pseudo-Convex Like' functions and Quasi-Convex-Like functions and described some of their properties. The class of all Strong Pseudo-Convex functions has been defined on a convex set S and it includes the class of all differentiable convex functions on S and is included in the class of all pseudo-convex functions on S. An interesting property of Strong Pseudo-Convex functions is that a local condition, such as the vanishing of the gradient, is a global optimality condition. One of the main results, that easily follows from [148 , Theorem 1] , is that the Kuhn-Tucker differential conditions are sufficient for optimality when in a M.P.P., the O.F. is strong pseudo-convex and the constraints are quasi-convex. Since the class of strong pseudo-convex functions is included in the class of pseudo-convex functions, therefore, they enjoy all other properties of pseudo-convex functions also. Another important work contained in this section is to provide an alternate proof for the property that for a pseudo-convex function every local minimum is a global minimum. Mangasarian [148] , proved this result by making use of the property that if a function is pseudo-convex then it is strictly quasi-convex also. The main feature of the alternate proof provided in the present thesis is that it does not make use of the property used by Mangasarian. As a consequence it follows that this property of every local minimum being a global minimum is possessed by pseudo-convex functions as their own right.

A fractional function under suitable assumptions is shown to be a strong pseudo-convex function.

Let S be a convex subset of an open set $D \subset R^n$, n-Euclidean space. Let $G \subset D$ be another set assumed to be non-convex. We now, under the assumption stated in Chapter I, define a function $f \in C^1$ to be

(a) Strong Pseudo-Convex (SPCX): over the convex set S, if for every X_1 and X_2 in S, $\exists$ a real, scalar, single valued function $K(X_1,X_2) > 0$, depending on the ordered pair (X_1,X_2) and on f, such that

$$(X_1 - X_2)' \nabla_x f(X_2) \leq K(X_1, X_2)\left[f(X_1) - f(X_2)\right]$$

(b) (i) Weakly Convex-Like (WCXL): over the set G, if for every X_1, $X_2 \in G$,

$$f(X_1) - f(X_2) \geqslant (X_1 - X_2)' \nabla_x f(X_2)$$

(ii) Weakly Strong Pseudo-Convex-Like (WSPCXL): over the set G, if for every X_1, X_2 in G, $\exists$ a real, scalar, single valued function $K(X_1, X_2) > 0$, depending on the ordered pair (X_1, X_2) and f such that

$$(X_1 - X_2)' \nabla_x f(X_2) \leq K(X_1, X_2)\left[f(X_1) - f(X_2)\right]$$

(iii) Quasi-Convex-Like (QXL): over the set G, if for all $X_1, X_2 \in G$

$$f(X_1) \leqslant f(X_2) \Rightarrow (X_1 - X_2)' \nabla_x f(X_2) \leqslant 0$$

or equivalently

$$(X_1 - X_2)' \nabla_x f(X_2) > 0 \Rightarrow f(X_1) \geqslant f(X_2)$$

Furthermore, we define a function f to be Strong Pseudo-Concave (SPCV), Weakly Concave-Like (WCVL), Weakly Strong Pseudo-Concave-Like (WSPCVL), and Quasi-Concave-Like (QVL) according as, -f, is SPCX, WCXL, WSPCXL, and QXL.

We shall relate now the four new types of convexity to the previously established notions of convexity, pseudo-convexity and quasi-convexity, and among themselves.

First of all we take up SPCX functions for discussion.

Property 1. If φ is CX on S, then φ is SPCX on S, but not conversely.

Proof. Since φ is CX on S, therefore for every X_1 and X_2 in S we have

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leqslant \varphi(X_1) - \varphi(X_2)$$

$$= K(X_1, X_2)\left[\varphi(X_1) - \varphi(X_2)\right]$$

where $K(X_1, X_2) = 1 > 0$. This precisely shows that φ is SPCX. The converse is not necessarily true can be seen from the example

$$\varphi(x_1, x_2) = \frac{2x_1 + 3x_2 + 1}{x_1 + 4x_2 + 2} \qquad x_1, x_2 \geqslant 0,\ x_1 \in R,\ x_2 \in R.$$

which is SPCX on R^2 but not convex[1].

Property 2. If φ is SPCX on S, then φ is PCX on S, but not conversely.

Proof. From the definition of SPCX function we obtain

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant 0 \implies \varphi(X_1) \geqslant \varphi(X_2)$$

for every X_1, X_2 in S. This precisely implies that φ is PCX.

---

1. Proof that a L.F.F. is SPCX, is available in the next section.

That the converse is not necessarily true can be seen from the example

$$\varphi(x) = \frac{c'x+\alpha}{d'x+\beta} + \left(\frac{c'x+\alpha}{d'x+\beta}\right)^3$$

for $d'X+\beta > 0$ over S and other symbols having the usual meanings.
Here φ is PCX (of course PCV also[1]) but not SPCX.

Remark 1. (i) From Property 2 above it thus follows that a SPCX function being PCX is also SQX, EQX and QX on S.

(ii) Since we have proved that if on S, φ is SPCX, it is PCX also, this implies from [148] that for a M.P.P in which the O.F. is SPCX and the constraints are QX, the Kuhn-Tucker differential conditions are sufficient for optimality, and that a strict converse duality theorem holds for such a M.P.P.

Now we take up WCXL functions for discussion of their properties.

Property 3. If φ be WCXL over G, then it is (WSPCXL) on G, but the converse is not necessarily true.

Proof. From the definition of WCXL function we obtain that

$$(x_1-x_2)'\nabla_x\varphi(x_2) \leq K(x_1,x_2)\left[\varphi(x_1)-\varphi(x_2)\right]$$

where $K(X_1, X_2) = 1 > 0$.

This implies that φ is (WSPCXL) on G. The converse is not necessarily true can be seen from the following example.

$$\varphi(x) = \frac{f(x)}{g(x)}$$

1. Proof, that φ is both PCX and PCV on S may be found, as a special case of a general theorem, in the last chapter of this thesis.

where f is WCXL and g is CVL on G and $f \geqslant 0$, $g > 0$, in which is (WSPCXL)[3] but not WCXL.

Property 4. If φ be (WSPCXL) on G then φ is PCX on G but the converse is not necessarily true.

Proof. From the definition of (WSPCXL) function we obtain

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant 0 \implies \varphi(X_1) \geqslant \varphi(X_2)$$

for every X_1, X_2 in G. The converse is not true may be seen from the following example.

Let
$$G = \{ x \; ; \; x \in R \, , \, x \neq 0 \}$$
$$\varphi(x) = x + x^3$$

Below we now state two results which can be easily proved.

Property 5. If $S \subset G$, then a (WSPCXL) functions on G is a SPCX function on S and a CXL function on G is a CX function on S.

~~Property 6. A SPCX function is CX on G if for every X_1, X_2, the function $K(X_1, X_2) \in \{0, 1\}$~~ .

Mangasarian [148] , for PCX functions defined on convex set, S, proved the following most important property. "Every local minimum of a PCX function, φ , over a convex set, S, is a global minimum also." To prove this property, Mangasarian, however, makes use of the property that, on a convex set every PCX function is SQX. Here an alternate proof, without using the property used by Mangasarian is provided.

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3. Proof, that φ is (WSPCXL) may be found in Theorem 6 of Section III of this chapter.

Proof. If possible let $X_o \in S$ be a local minimum and $X_* (\neq X_o) \in S$ be a global minimum of φ over S, such that

$$\varphi(X_*) < \varphi(X_o) \qquad (4.2.1)$$

Since the function φ is PCX, therefore, (4.2.1) implies that

$$(X_* - X_o)' \nabla_x \varphi(X_o) < 0$$

i.e. $$[\nabla_x \varphi(X_o)]' (X_* - X_o) < 0 \qquad (4.2.2)$$

(4.2.2) implies that there exists a direction $X_* - X_o$ at X_o along which the function φ decreases in the ε-neighbourhood $N_\varepsilon(X_o)$ of X_o. The set S being convex and X_*, X_o being in S, imply that we can have a point $X_\lambda = X_o + \lambda (X_* - X_o)$ belonging to the set $N_\varepsilon(X_o) \cap S$ for however small $\lambda > 0$, such that

$$\varphi(X_\lambda) < \varphi(X_o)$$

which is a contradiction to the fact that X_o is a local minimum. Hence the result follows.

SECTION - III

NATURE OF PRODUCTS, QUOTIENTS, RATIONAL POWERS AND COMPOSITION OF CONVEX-LIKE FUNCTIONS:

In this section we shall study systematically the nature of products, quotients and rational powers and compositions of convex-like functions. For convenience we note that throughout this chapter, S is to denote a convex subset of R^n and G a non-convex subset of R^n.

PRODUCT OF FUNCTIONS

Theorem 1. If f and g be CV and non-negative functions over S, then fg is EQV over S.

Proof. Let $\varphi(X) = f(X)g(X)$ for $X \in S$.

Without any loss of generality we assume that for any $X_1, X_2 \in S$ satisfying, $\varphi(X_1) \neq \varphi(X_2)$, we have,

$$\varphi(X_2) = \mathrm{Min}\left[\varphi(X_1), \varphi(X_2)\right] \tag{4.3.1}$$

Now for all $\lambda \in]0,1[$,

$$\varphi(X_\lambda) = f[\lambda X_1 + (1-\lambda)X_2]\, g[\lambda X_1 + (1-\lambda)X_2]$$

where $X_\lambda = \lambda X_1 + (1-\lambda) X_2$, for all $\lambda \in]0,1[$.

Using concavity and non-negativity of both f and g we obtain

$$\varphi(X_\lambda) \geq \left[\lambda f(X_1) + (1-\lambda) f(X_2)\right]\left[\lambda g(X_1) + (1-\lambda) g(X_2)\right]$$

$$= \lambda^2 \varphi(X_1) + (1-\lambda)^2 \varphi(X_2) + \lambda(1-\lambda)\left[f(X_1)g(X_2) + f(X_2)g(X_1)\right] \tag{4.3.2}$$

Using (4.3.1) we obtain from (4.3.2):

$$\varphi(X_\lambda) - \varphi(X_2) \geq \lambda(1-\lambda) E \tag{4.3.3}$$

where $E = f(X_1)g(X_2) + f(X_2)g(X_1) - 2 f(X_2)g(X_2)$

In view of (4.3.1) we consider the following cases only.

$$\left.\begin{array}{l}
\text{(i)}\ \left.\begin{array}{l} f(X_1) > f(X_2),\ g(X_1) > g(X_2);\ f(X_1) \geq f(X_2),\ g(X_1) > g(X_2); \\ \quad f(X_1) > f(X_2),\ g(X_1) \geq g(X_2). \end{array}\right] \\
\text{(ii)}\ \left.\begin{array}{l} f(X_1) > f(X_2),\ g(X_1) < g(X_2);\ f(X_1) \geq f(X_2),\ g(X_1) < g(X_2); \\ \quad f(X_1) > f(X_2),\ g(X_1) \leq g(X_2). \end{array}\right] \\
\text{(iii)}\ \left.\begin{array}{l} f(X_1) < f(X_2),\ g(X_1) > g(X_2);\ f(X_1) \leq f(X_2),\ g(X_1) > g(X_2); \\ \quad f(X_1) < f(X_2),\ g(X_1) \geq g(X_2) \end{array}\right]
\end{array}\right\} \tag{4.3.4}$$

Now

$$E = g(x_2)\left[f(x_1) - f(x_2)\right] + f(x_2)\left[g(x_1) - g(x_2)\right] \tag{4.3.5}$$

Again taking (4.3.1) into consideration we have

$$E > f(x_1)\,g(x_2) + f(x_2)\,g(x_1) - f(x_2)g(x_2) - f(x_1)\,g(x_1)$$

$$= \left[f(x_1) - f(x_2)\right]\left[g(x_2) - g(x_1)\right] \tag{4.3.6}$$

In view of (4.3.4), (4.3.5), (4.3.6) and the non-negativity of both f and g over S, we assert that $E \geqslant 0$ or $E > 0$.

Thus, from (4.3.3) we obtain $\quad \varphi(x_3) > \varphi(x_2) \quad$ (4.3.7)

Hence (4.3.1) and (4.3.7) yield

$$\varphi(x_3) > \mathrm{Min}\left[\varphi(x_1), \varphi(x_2)\right]$$

i.e. φ is EQV on S.

Remark 1. The function fg which is proved to be EQV in Theorem 1 above, can be shown, under the conditions of Theorem 1 to be QV also.

Remark 2. We can prove similarly the following more results tabulated below in Table 1.

TABLE - 1

f	SV >0	CV $\geqslant 0$	SV >0	CX $\leqslant 0$	SX <0	CX $\leqslant 0$	SX <0	CV $\geqslant 0$	CV $\geqslant 0$	SV >0	SV >0	CX $\leqslant 0$	CX $\leqslant 0$	SX <0	SX <0
g	CV $\geqslant 0$	SV >0	SV >0	CX $\leqslant 0$	CX $\leqslant 0$	SX <0	SX <0	CX $\leqslant 0$	SX <0	CX $\leqslant 0$	SX <0	CV $\geqslant 0$	SV >0	CV $\geqslant 0$	SV >0
fg	EQV	EQV	EQV	EQV	EQV	EQV	EQV	EQX	EQX	EQX	EQX	EQX	EQX	EQX	EQX

Theorem 2. If $f_j \in C^1$, $j = 1,2,\ldots,p$ (p being finite) are strictly positive and concave functions on the set S, then the function φ defined by,

(i) $\varphi = \prod_{j=1}^{p} f_j$

(ii) $\varphi = \prod_{j=1}^{p} [f_j]^{\delta_j}$, where, δ_j, for j=1,2,---,p is a non-negative integer,

(iii) $\varphi = \prod_{j=1}^{p} [f_j]^{\frac{1}{\gamma_j}}$, where, for j=1,2,---,p (i) γ_j is a positive integer, and (ii) $[f_j]^{\frac{1}{\gamma_j}}$ is strictly positive over S,

(iv) $\varphi = \prod_{j=1}^{p} [f_j]^{m_j/n_j}$, where, (i) m_j, n_j; j=1,2,---,p; are respectively non-negative and positive integers, and (ii) each $[f_j]^{1/n_j}$ is strictly positive over S,

is PCV over S.

Proof. We here provide the proof for part (i). Other parts can be proved similarly.

Let for any $X \in S$,

$$\varphi(X) = \prod_{j=1}^{p} f_j(X) \tag{4.3.8}$$

$$\Rightarrow \quad \nabla_x \varphi(X_2) = \left[\sum_{i=1}^{p} \nabla_x f_i(X_2) \right] \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j(X_2)$$

$$\Rightarrow \quad (X_1 - X_2)' \nabla_x \varphi(X_2) = \left[\sum_{i=1}^{p} (X_1 - X_2)' \nabla_x f_i(X_2) \right] \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j(X_2)$$

Using the concavity and strict positivity of f_j's over S we obtain for every $X_1, X_2 \in S$,

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant \left\{ \sum_{i=1}^{p} \left[f_i(X_1) - f_i(X_2) \right] \right\} \prod_{\substack{j=1 \\ j \neq i}}^{p} f_j(X_2)$$

$$= \varphi(X_2) \left[\sum_{j=1}^{p} \frac{f_j(X_1)}{f_j(X_2)} - p \right]$$

$$= \varphi(X_2) \left[\log \varphi(X_1) - \log \varphi(X_2) \right]$$

$$+ \varphi(X_2) \left[\sum_{j=1}^{p} u_j - \sum_{j=1}^{p} \log u_j - p \right] \qquad (4.3.9)$$

where $\qquad u_j = \dfrac{f_j(X_1)}{f_j(X_2)}, \quad j = 1, 2, \ldots, p.$

Using Lemma 3 of Chapter II we obtain from (4.3.9),

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant \varphi(X_2) \left[\log \varphi(X_1) - \log \varphi(X_2) \right] \qquad (4.3.10)$$

From which we obtain,

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leqslant 0 \implies \log \varphi(X_1) \leqslant \log \varphi(X_2)$$

$$\implies \varphi(X_1) \leqslant \varphi(X_2)$$

i.e. φ is a PCV function on S.

Remark 3. It can be easily seen from (4.3.10) that the function φ defined in Theorem 2 above is QV also.

We now state two results in the form of corollaries, which are immediate deductions from Theorem 2.

Corollary 1. If (i) $C_j = (c_{j1}, c_{j2}, \text{-----}, c_{jn})'$ is an nx1 column vector for j=1,2,-----,p (p being finite); (ii) α_j, j=1,2,-----,p is an arbitrary scalar constant, then the function φ given by

$$\varphi(X) = (C_1'X+\alpha_1)(C_2'X+\alpha_2) \text{ ----- } (C_p'X+\alpha_p) \qquad (4.3.11)$$

is PCV (QV) over S, provided over S, $C_j'X+\alpha_j > 0$, for j=1,2,---,p;

Corollary 2 . In Theorem 2, if $f_1 = f$, $f_2 = g$, $f_j = 1$ for j=3,4,-----,p, then φ = fg is PCV(QV) over S.

Remark 4. For the N.L.I.P. φ = fg in Corollary 2 above, the following more results given in the Table 2 below, can be proved easily.

TABLE 2

f	C^1	CV >0	CX <0	CX <0
g	C^1	CX <0	CV >0	CX <0
$f \cdot g$		PCX	PCX	PCV

Remark 5. Let for $X \in S$,

$$f(X) = c'X - (X'GX)^{1/2} + \alpha$$

$$g(X) = d'X - (X'HX)^{1/2} + \beta$$

where c,d are nx1 column vectors, α, β are arbitrary scalar constants and G and H are real symmetric positive semi-definite matrices, each

of order $m \times n$, such that f and g are well defined concave functions over S, assumed to be strictly positive throughout S.

It may now be observed here that both f and $g \notin C^1$ over S and fg is EQV but not necessarily PCV.

Similarly if we have at least one of f and g; $\notin C^1$ over S, we will see that fg is EQV but not necessarily PCV. But if we let,

$$f(x) = c'x + \alpha$$

$$g(x) = d'x + \beta$$

where c, d, α and β are as defined above, and $c'x + \alpha > 0$, $d'x + \beta > 0$ over S. Applying the results proved above, here we confirm that fg is PCV also.

Below we now state (without proof) a theorem which asserts that if each of the concave functions f_j in Theorem 2 above be replaced by a WCVL function defined on G, then φ given by (i), (ii), (iii) and (iv) is PCV over G.

Theorem 3. If $f_j \in C^1$, $j = 1, 2, \text{------}, p$ (p being finite) are strictly positive and WCVL functions defined on the set G, then the function φ defined by,

(i) $\varphi = \prod_{j=1}^{p} f_j$

(ii) $\varphi = \prod_{j=1}^{p} [f_j]^{\delta_j}$, where δ_j for $j = 1, 2, \text{------}, p$ is a non-negative integer.

(iii) $\varphi = \prod_{j=1}^{p} [f_j]^{1/\gamma_j}$, where for $j = 1, 2, \text{------}, p$; (i) $\gamma_j > 0$ is an integer, (ii) $[f_j]^{1/\gamma_j} > 0$ over G.

(iv) $\varphi = \prod_{j=1}^{p} [f_j]^{m_j/n_j}$ where, for j=1,2,------,p (i) m_j, n_j are respectively non-negative and positive integers, and

(ii) $[f_j]^{1/n_j}$ is strictly positive over G,

is PCV over G.

QUOTIENT OF TWO FUNCTIONS

Theorem 4. If, (i) f be a CX(CV) and non-negative function over S, (ii) g be a CV(CX) and strictly positive function over S, then f/g is an EQX(EQV) function over S.

Proof. Let $\varphi(X) = f(X)/g(X)$ for $X \in S$. (4.3.12)

where it is assumed that f is CX and g is CV over S.

Without loss of generality we assume that for any X_1, $X_2 \in S$ satisfying $\varphi(X_1) \neq \varphi(X_2)$ we have

$$\varphi(X_2) = \mathrm{Max}[\varphi(X_1), \varphi(X_2)] \tag{4.3.13}$$

Let for all $\lambda \in]0,1[$,

$$X_\lambda = \lambda X_1 + (1-\lambda) X_2$$

Then we have

$$\varphi(X_\lambda) - \varphi(X_2) = \frac{f[\lambda X_1 + (1-\lambda) X_2]}{g[\lambda X_1 + (1-\lambda) X_2]} - \frac{f(X_2)}{g(X_2)}$$

$$= \frac{g(X_2)\, f[\lambda X_1 + (1-\lambda) X_2] - f(X_2)\, g[\lambda X_1 + (1-\lambda) X_2]}{g(X_2)\, g[\lambda X_1 + (1-\lambda) X_2]} \tag{4.3.14}$$

Using the convexity and non-negativity of f, and concavity and strict positivity of g over S, we obtain from (4.3.14),

$$\varphi(x_\lambda) - \varphi(x_2) \leqslant \frac{g(x_2)[\lambda f(x_1) + (1-\lambda) f(x_2)] - f(x_2)[\lambda g(x_1) + (1-\lambda) g(x_2)]}{g(x_2)\, g[\lambda x_1 + (1-\lambda) x_2]}$$

$$= \frac{\lambda g(x_1)}{g[\lambda x_1 + (1-\lambda) x_2]} [\varphi(x_1) - \varphi(x_2)] \qquad (4.3.15)$$

Combining (4.3.13) and (4.3.15) we obtain for $x_\lambda = \lambda x_1 + (1-\lambda) x_2$ for all $\lambda \in]0,1[$, $\varphi(x_1) \neq \varphi(x_2)$

$$\varphi(x_\lambda) < \text{Maximum}\, [\varphi(x_1), \varphi(x_2)]$$

i.e. φ is EQX over S.

Similarly, when f is CV and non-negative over S and g is CX and strictly positive over S, it can be proved that $\varphi = f/g$ is EQV on S.

Remark 6. (i) It is very important to note here that if g be a strictly positive linear function defined on S, then in Theorem 4 above, the non-negativity restriction on f is not necessary. Another result analogous to Theorem 4, is as follows:

" If f be a CX(CV) function defined on S and g be a strictly positive linear function defined on S, then f/g is EQX(EQV) over S."

(ii) As a very special case now if over the set S, we assume f to be linear such that f is both CX as well as CV on S, and g to be strictly positive linear function, then from the above stated result it follows that f/g is both EQX and EQV on S, i.e. a L.F.F. is, in this case, an

E M function on S, whereas Bela Martos [151], proves it to be QX.

Remark 7. It can be easily proved now that if f be non-negative and CX(CV) over S, g be strictly positive and CV(CX) over S(if g be linear, then condition of non-negativity on f is not necessary) then f/g is QV over S.

Remark 8. We now record the following more results given in the Table 3 below, which can either be deduced from above or else proved in a similar way.

TABLE - 3

f	CX ≥ 0	SX >0	SX >0	CV ≥ 0	SV >0	SV >0	CX ≤ 0	CX ≤ 0	SX <0	SX <0	CX ≥ 0	CX ≥ 0	CX >0	SX >0	CV ≥ 0
g	SV >0	CV >0	SV >0	SX >0	CX >0	SX >0	CX >0	SX >0	CX >0	CX >0	CX <0	CX <0	CX <0	SX <0	CV <0
f/g	EQX	EQX	EQX	EQX	EQX	EQX	EQX	EQX	EQX	EQX	EQV	EQV	EQV	EQV	EQX

TABLE-3-Continued

f	CV ≥ 0	SV >0	CV ≥ 0	CV ≤ 0	CV ≤ 0	SV <0	SV <0	CX ≤ 0	CV ≤ 0	SX <0	CX ≤ 0	SX <0	SV <0	CV ≤ 0	SV <0
g	SV <0	CV <0	CV <0	CV >0	SV >0	CV >0	CV >0	CV <0	CX <0	CV <0	SV <0	SV <0	CX <0	SX <0	SX <0
f/g	EQX	EQX	EQX	EQV	EQV	EQV	EQV	EQV	EQX	EQV	EQV	EQV	EQX	EQX	EQX

Theorem 5. If, over the set S, $f \in C^1$ be a CX(CV) and non-negative function, and $g \in C^1$ be a CV(CX) and strictly positive function, then f/g is a SPCX(SPCV) function over S.

Proof. Let $\varphi(X) = \dfrac{f(X)}{g(X)}$ for $X \in S$, (4.3.16)

where we assume that over S, f is CX and non-negative and g is CV and strictly positive.

For any two points $X_1, X_2 \in S$ we have from (4.3.16),

$$(X_1 - X_2)' \nabla_X \varphi(X_2) = \frac{1}{[g(X_2)]^2}\left[g(X_2)(X_1-X_2)'\nabla_X f(X_2) - f(X_2)(X_1-X_2)'\nabla_X g(X_2)\right] \quad (4.3.17)$$

Using convexity and non-negativity of f and strict positivity of g over S, we obtain from (4.3.17)

$$(X_1-X_2)'\nabla_X \varphi(X_2) \leq \frac{1}{[g(X_2)]^2}\left\{g(X_2)[f(X_1)-f(X_2)] - f(X_2)[g(X_1)-g(X_2)]\right\}$$

$$= \frac{g(X_1)}{g(X_2)}\left[\varphi(X_1) - \varphi(X_2)\right]$$

$$= K(X_1,X_2)\left[\varphi(X_1) - \varphi(X_2)\right] \quad (4.3.18)$$

where $K(X_1,X_2) = \frac{g(X_1)}{g(X_2)} > 0$, depends upon the ordered pair (X_1,X_2) and the function φ. Hence the function φ is SPCX.

Similarly, if we assume f be CV and non-negative and g to be CX and strictly positive, it is easily proved that f/g is SPCV on S.

Remark 9. It is interesting to remark here that if $g \in C^1$ be a strictly positive linear function on S, then in Theorem 5 above, the non-negativity restriction on f is not necessary. Thus corresponding to Theorem 5 we have the result as follows:

"If $f \in C^1$ be a CX(CV) function defined on S and g be a strictly positive linear function defined on S, then f/g is SPCX(SPCV) over S".

(ii) As a very special case now if over the set S, we assume f to be linear such that f is both CX and CV and also $f \in C^1$, and we assume g

to be strictly positive linear, then $g \in C^1$ and is both CV and CX, then we deduce that f/g is both SPCX and SPCV i.e. in this case the L.F.F. f/g belongs to the class of SPCM functions.

Remark 10. The following results tabulated below in Table 4 can now be proved easily.

$f \in C^1$	CV $\geqslant 0$	CX $\leqslant 0$	CX $\geqslant 0$	CV $\leqslant 0$	CX $\leqslant 0$	CV $\geqslant 0$	CV $\leqslant 0$
$g \in C^1$	CX > 0	CV < 0	CX < 0	CV > 0	CX > 0	CV < 0	CX < 0
f/g	SPCV	SPCV	SPCV	SPCV	SPCX	SPCX	SPCX

Remark 11. Let for $X \in S$,

$$f(X) = c'X + (X'GX)^{1/2} + \alpha$$
$$g(X) = d'X - (X'HX)^{1/2} + \beta$$

Where symbols have the same meanings as in Remark 5, such that $f(X)$ is CX assumed to be non-negative and g is CV assumed to be strictly positive over S.

It may now be observed that f as well as $g \notin C^1$ over S and f/g is EQX but not necessarily SPCX function over S. Similarly if we have at least one of f and $g \notin C^1$ over S, we will observe that f/g is EQX but not necessarily SPCX over S. But if we let,

$$f(X) = c'X + X'GX + \alpha$$
$$g(X) = d'X - X'HX + \beta$$

where symbols are as already explained such that f is CX and assumed to be non-negative over S and g is CV and assumed to be strictly positive over S. Applying the results proved above, here we confirm that f/g is SPCX also.

Remark 12. We now state a theorem proof of which follows on the lines of the proof provided for Theorem 5.

Theorem 6. If over the set G, f be a non-negative WCXL(WCVL) function and g be a strictly positive WCVL(WCXL) function, then f/g is a WSPCXL(WSPCVL) function.

RATIONAL POWERS OF A FUNCTION:

We know that if f be a strictly positive CX function, then $\frac{1}{f}$ is not necessarily a CV function. But below we prove that if f be a strictly positive EQX(PCX) function then $\frac{1}{f}$ is EQV(PCV).

Theorem 7. If f is an EQX(EQV) and strictly positive function defined over the set S, then $\frac{1}{f}$ is an EQV(EQX) function over S.

Proof. Let f be EQX in S, and let

$$\varphi(x) = \frac{1}{f(x)} \qquad \text{for} \quad x \in S.$$

Since f is EQX on S, therefore for $x_\lambda = \lambda x_1 + (1-\lambda) x_2$, for all $\lambda \in]0,1[$, and every $x_1, x_2 \in S$ satisfying $f(x_1) \neq f(x_2)$, we have

$$f(x_\lambda) < \text{Max}\left[f(x_1), f(x_2)\right] \tag{4.3.19}$$

Without loss of generality we assume that

$$f(x_2) = \text{Max}\left[f(x_1), f(x_2)\right] \tag{4.3.20}$$

$$\Rightarrow f(X_1) < f(X_2) \quad i.e. \quad \frac{1}{f(X_2)} < \frac{1}{f(X_1)}$$

$$\Rightarrow \qquad \varphi(X_2) = \mathrm{Min}\,[\varphi(X_1), \varphi(X_2)] \qquad (4.3.21)$$

(4.3.19) and (4.3.20) yield,

$$\varphi(X_\lambda) > \varphi(X_2) \qquad (4.3.22)$$

From (4.3.21) and (4.3.22) we obtain that

$$\varphi(X_\lambda) > \mathrm{Min}\,[\varphi(X_1), \varphi(X_2)]$$

i.e. $\varphi = \frac{1}{f}$ is EQV on S.

When f is EQV, then it similarly follows that $\frac{1}{f}$ is EQX.

Remark 13. A result parallel to Theorem 7 can be easily established for QX(QV) functions.

Theorem 8. If $f \in C^1$ and is PCX(PCV) and strictly positive over G, then $\frac{1}{f}$ is PCV(PCX) over G.

Proof. Let f be PCX over G. Therefore for every X_1, X_2 in G we have

$$(X_1 - X_2)' \nabla_x f(X_2) \geq 0 \Rightarrow f(X_1) \geq f(X_2) \qquad (4.3.23)$$

Let $\qquad \varphi(X) = \frac{1}{f(X)} \qquad$ for $\quad X \in G$

Therefore, $(X_1 - X_2)' \nabla_x \varphi(X_2) = \frac{-1}{[f(X_2)]^2} (X_1 - X_2)' \nabla_x f(X_2) \qquad (4.3.24)$

From (4.3.24) we obtain

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leq 0 \Rightarrow (X_1 - X_2)' \nabla_x f(X_2) \geq 0$$

$$\Rightarrow f(X_1) \geq f(X_2)$$

$$\Rightarrow \varphi(X_1) \leq \varphi(X_2)$$

strictly positive or strictly negative over G, n is any positive integer.

Proof. Let $\varphi(X) = [f(X)]^{1/n}$, $X \in G$, be strictly positive over G. Then for every $X_1, X_2 \in G$ and $\varphi \in C'$,

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant 0 \iff \frac{(X_1 - X_2)' \nabla_x f(X_2)}{n \, [f(X_2)]^{1-\frac{1}{n}}} \geqslant 0$$

$$\iff (X_1 - X_2)' \nabla_x f(X_2) \geqslant 0$$

$$\implies f(X_1) \geqslant f(X_2)$$

$$\implies f^{1/n}(X_1) \geqslant f^{1/n}(X_2)$$

$$\implies \varphi(X_1) \geqslant \varphi(X_2)$$

i.e. $\varphi = f^{1/n}$ is PCX over G. Similarly when $f^{1/n}$ is strictly negative over G, it is PCV over G.

Below we now state a few results which can be easily proved.

1. If f is an EQV(QV) and non-negative function over S, then $f^{1/n}$ is EQV(QV) or EQX(QX) according as $f^{1/n}$ is strictly positive (non-negative) or strictly negative (non-positive) over S, n is any positive integer.

2. If $f \in C^1$ is a PCX(PCV) and strictly positive function defined over G, then $f^{1/n}$ is PCX or PCV (PCV or PCX) over G according as $f^{1/n}$ is strictly positive or strictly negative

3. If f is an EQX or EQV (QX or QV) and non-negative function over S, then f^n is EQX or EQV (QX or QV) over S for any non-negative integer n.

4. If $f \in C^1$ is PCX(PCV) and non-negative function over G, then f^n is PCX(PCV) over G for any non-negative integer.

Below we now state certain corollaries to above stated results.

Corollary 1. If f is an EQX(QX) and non-negative function over S, then $f^{p/q}$ is EQX(QX) or EQV(QV) according as $f^{1/q}$ is strictly positive (non-negative) or strictly-negative (non-positive) over S, where p and q are respectively any non-negative and positive integers.

Corollary 2. If f is an EQV(QV) and non-negative function over S, then $f^{p/q}$ is EQV(QV) or EQX(QX) over S according as $f^{1/q}$ is strictly positive (non-negative) or strictly negative (non-positive) over S, where p and q are, respectively, non-negative and positive integers.

Corollary 3. If $f \in C^1$ is a PCX and strictly positive function over G, then $f^{p/q}$ is PCX or PCV over G, according as $f^{1/q}$ is strictly positive or strictly negative over G, where p and q as in Corollary 1.

Corollary 4. If $f \in C^1$ is a PCV and strictly positive function over G, then $f^{p/q}$ is PCV or PCX over G, according as $f^{1/q}$ is strictly positive or strictly negative over G, p and q being as in Corollary 1.

COMPOSITE FUNCTIONS:

Below we now prove two theorems, from which the results from Theorem 7 onwards upto Corollary 4 above, are deducible as particular cases. It is, however, to point out here that the amount of labour involved both ways appears to be equal. Also in convex functions, it is a well known result (see for example [41,42, 160]) that if f is be a CX function defined on a convex set $S \subset R^n$, and g is a non-decreasing real valued CX function on an interval $I \subset R$ such that $f(S) \subset I$, then the function $g \circ f$ is CX on S. The following two

theorems generalize the above result to more general classes of PCX, EQX and QX functions, with the restriction, pseudo-convexity, explicit quasi-convexity and quasi-convexity on f but without any such convexity-like restriction on g. The results for the concepts of pseudo-concavity, explicit quasi-concavity and quasi-concavity used on f are analogous.

Theorem 11. If f is EQX(QX) on the set S and g is a non-decreasing real valued function on an interval $I \subset R$ such that $f(S) \subset I$, then the function $g \circ f$ is EQX(QX) on S.

Proof. Let us assume that f is EQX on S. Therefore, for every $X_1, X_2 \in S$, satisfying

$$f(x_1) \neq f(x_2) \tag{4.3.25}$$

and for $\quad x_\lambda = \lambda x_1 + (1-\lambda) x_2 \quad$ for all $\lambda \in \,]0,1[$,

$$f(x_\lambda) < \operatorname{Max}\left(f(x_1), f(x_2)\right) \tag{4.3.26}$$

Since g is given to be a non-decreasing function on $I \subset R$, therefore (4.3.25) and (4.3.26) yield,

$$g[f(x_1)] \neq g[f(x_2)]$$

and

$$g[f(x_\lambda)] < \operatorname{Max}\left[g(f(x_1)), g(f(x_2))\right]$$

i.e.

$$(g \circ f)(x_1) \neq (g \circ f)(x_2)$$

and

$$(g \circ f)(x_\lambda) < \operatorname{Max}\left[(g \circ f)(x_1), (g \circ f)(x_2)\right]$$

Thus the function $g \circ f$ is EQX.

Now if we assume that f is QX on S, then we similarly obtain that $g \circ f$ is QX over S. This part of the theorem is also proved in Arrow and Enthoven [8].

Theorem 12. If f is PCX over the set G and g is a differentiable real valued function with derivative strictly positive on an interval $I \subset R$, such that $f(G) \subset I$, then the function $g \circ f$ is PCX over G.

Proof. Since f is PCX on G, therefore for every X_1, X_2 in G,

$$(X_1 - X_2)' \nabla_x f(X_2) \geqslant 0 \implies f(X_1) \geqslant f(X_2) \tag{4.3.27}$$

Now
$$(X_1 - X_2)' \nabla_x (g \circ f)(X_2) = \frac{\partial g}{\partial y} \left[(X_1 - X_2)' \nabla_x f(X_2) \right] \tag{4.3.28}$$

where $y = f(X)$ for all $X \in G$.

On using the given conditions on g, from (4.3.28) we obtain,

$$\begin{aligned} (X_1 - X_2)' \nabla_x (g \circ f)(X_2) \geqslant 0 &\iff (X_1 - X_2)' \nabla_x f(X_2) \geqslant 0 \\ &\implies f(X_1) \geqslant f(X_2) \\ &\implies (g \circ f)(X_1) \geqslant (g \circ f)(X_2) \end{aligned}$$

i.e. $g \circ f$ is PCX on G.

Berge [41] and Berge and Houri [42] prove the following theorem.

If $f_1, f_2, \text{------}, f_p$ are continuous CX(CV) mappings of S into $I_1, I_2, \text{------}, I_p$ respectively, $I_j \subset R$, $j = 1, 2, \text{------}, p$ and if g is a non-decreasing continuous CX(CV) mapping of $\prod_{i=1}^{p} f_i(S) \subset R^p$ into $I \subset R$, then the mapping φ of S into I defined by

$$\varphi(X) = g[f_1(X), f_2(X), \text{------} f_p(X)]$$

is CX(CV).

Berge [41], extended the above result to the case in which the convexity(concavity) of the function g in the above result is replaced by quasi-convexity (quasi-concavity) thus giving the result that the function φ, defined in above, is QX(QV).

In the present work we have replaced the convexity(concavity) of the function g by explicit quasi-convexity (explicit quasi-concavity) and pseudo-convexity (pseudo-concavity) and improved upon the results obtained by Berge [41] and Berge and Houri [42], as follows. It is important to remark over here that many theorems proved on products and quotients of functions in this section, can also be obtained as particular cases to the following two theorems.

Theorem 13. If for i=1,2,----,p all functions f_i be CX(CV) on the set S with $f_i : S \rightarrow I_i \subset R$ and if g, be a non-decreasing EQX(EQV) function on the set D with $g: D \rightarrow I \subset R$, $D = \prod_{i=1}^{p} I_i \subset R^p$ then the function φ given by

$$\varphi(X) = g[f_1(X), f_2(X), ----, f_p(X)]$$

is EQX(EQV) on S.

Proof. Let X_1, X_2 be any two points in S and let

$$X_\lambda = \lambda X_1 + (1-\lambda) X_2 \qquad \text{for all} \quad \lambda \in \,]0,1[.$$

If we assume that for any $X \in S$,

$$Y = (f_1(X), f_2(X), ----, f_p(X))'$$

then $Y \in D$ and corresponding to X_1, X_2 in S we can write any two points Y_1, Y_2 in D as,

$$Y_1 = \left(f_1(x_1), f_2(x_1), \text{----}, f_p(x_1)\right)'$$

$$Y_2 = \left(f_1(x_2), f_2(x_2), \text{-----}, f_p(x_2)\right)'$$

Let g be EQX on D. Therefore, for $g(Y_1) \neq g(Y_2)$

we have

$$g(Y_\lambda) < \mathrm{Max}\left[g(Y_1), g(Y_2)\right] \tag{4.3.29}$$

Now we have,

$$\varphi(x_\lambda) = g\left[f_1(x_\lambda), f_2(x_\lambda), \text{-----}, f_p(x_\lambda)\right] \tag{4.3.30}$$

Using the convexity all f_i, $i=1,2,$------$,p$ on S and the fact that

g is non-decreasing, we obtain from (4.3.30),

$$\varphi(x_\lambda) \leq g\left[\lambda f_1(x_1) + (1-\lambda) f_1(x_2), \text{-----}, \lambda f_p(x_1) + (1-\lambda) f_p(x_2)\right]$$

$$= g\left[\lambda\left(f_1(x_1), f_2(x_1), \text{----}, f_p(x_1)\right) + (1-\lambda)\left(f_1(x_2), f_2(x_2), \text{----}, f_p(x_p)\right)\right]$$

$$= g\left[\lambda Y_1 + (1-\lambda) Y_2\right] \tag{4.3.31}$$

(4.3.29) and (4.3.31) yield

$$\varphi(x_\lambda) < \mathrm{Max}\left[g(Y_1), g(Y_2)\right]. \tag{4.3.32}$$

Since $\qquad g(Y_1) \neq g(Y_2) \iff \varphi(x_1) \neq \varphi(x_2)$

and $\qquad \mathrm{Max}\left[g(Y_1), g(Y_2)\right] \iff \mathrm{Max}\left[\varphi(x_1), \varphi(x_2)\right] \qquad (4.3.33)$

From (4.3.33) and (4.3.32) we conclude that φ is EQX on S.

A similar proof holds when g is EQV and all f_i are concave functions.

Theorem 14. If for $i = 1,2,$------$,p$, all functions $f_i \in C^1$ be

CX(CV) on the set S with $f_i: S \to I_i \subset R$, and if $g, \in C^1$, be an increasing PCX(PCV) function on the set $D = \prod_{i=1}^{p} I_i \subset R^p$ with $g: D \to I \subset R$, then the function φ given by

$$\varphi(X) = g\left[f_1(X), f_2(X), ----, f_p(X)\right]$$

is PCX(PCV) on S.

Proof. Let $Y \in D$ for $X \in S$, such that

$$Y = \left(f_1(X), f_2(X), ----, f_p(X)\right)' \tag{4.3.34}$$

and

$$\left.\begin{aligned} Y_1 &= \left(f_1(X_1), f_2(X_1), ----, f_p(X_1)\right)' \\ Y_2 &= \left(f_1(X_2), f_2(X_2), ----, f_p(X_2)\right)' \end{aligned}\right\} \tag{4.3.35}$$

for $X_1, X_2 \in S$, so that

$$\varphi(X_1) - \varphi(X_2) = g(Y_1) - g(Y_2) \tag{4.3.36}$$

Now

$$\nabla_x \varphi(X) = \left[\nabla_x f_1(X), \nabla_x f_2(X), -----, \nabla_x f_p(X)\right]\left[\nabla_Y g(Y)\right]$$

$$\Rightarrow (X_1 - X_2)'\nabla_x \varphi(X_2) = \left[(X_1 - X_2)'\nabla_x f_1(X_2), ----, (X_1 - X_2)'\nabla_x f_p(X_2)\right]\left[\nabla_Y g(Y_2)\right] \tag{4.3.37}$$

Using the facts that f_i, i=1,2,------,p are CX on S and g is increasing on D, (4.3.37) yields,

$$(X_1 - X_2)'\nabla_x \varphi(X_2) \leq \left[f_1(X_1) - f_1(X_2), ----, f_p(X_1) - f_p(X_2)\right]\left[\nabla_Y g(Y_2)\right] \tag{4.3.38}$$

(4.3.38) and (4.3.35) give,

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leq (Y_1 - Y_2)' \nabla_Y g(Y_2)$$

Therefore, $(X_1 - X_2)' \nabla_X \varphi(X_2) \geqslant 0 \Rightarrow (Y_1 - Y_2)' \nabla_Y g(Y_2) \geqslant 0$

$$\Rightarrow g(Y_1) \geqslant g(Y_2) \quad [\text{Since } g \text{ is PCX}]$$

$$\Rightarrow \varphi(X_1) \geqslant \varphi(X_2) \quad [\text{From } (4\cdot 3\cdot 36)]$$

i.e. φ is PCX on S.

A similar proof holds if we assume all f_i to be CV and g to be PCV.

Remark 14. Below we give a theorem (without proof) which is slightly more general than Theorem 14 and is based on concept that for all $i=1,2,\text{------},p$, f_i are WCXL (WCVL) functions. Proof of the theorem follows on the lines that of Theorem 14.

Theorem 15. If for $i=1,2,\text{------},p$, all functions $f_i \in C^1$ be WCXL(WCVL) on the set G (of which the convex set S be assumed to be a sub-set, for the purpose of this theorem) with $f_i : G \rightarrow I_i \subset R$ and if $g, \in C^1$, be an increasing PCX(PCV) function on the set $D = \prod_{i=1}^{p} I_i \subset R^p$ with $g: D \rightarrow I \subset R$, then the function φ given by

$$\varphi(X) = g\left[f_1(X), f_2(X), \text{-----}, f_p(X)\right]$$

is PCX(PCV) on G.

Remark 15. In Theorem 15 if we replace the condition of pseudo-convexity (concavity) on the function g by weakly strong pseudo-convexity (concavity) like condition, then we obtain that φ is also a 'weakly strong pseudo-convex (pseudo-concave) like function on G. Also if the set G be replaced by convex set S and the condition of pseudo-convexity (concavity) on g be replaced by strong

pseudo-convexity (concavity) then φ is a strong pseudo-convex (concave) function.

We now provide results which connect strong-pseudo-convexity, pseudo-convexity and explicit quasi-convexity of a function defined on a convex subset of R^n with the strong pseudo-convexity, pseudo-convexity and explicit quasi-convexity of another function defined on a subset of R^2 from which the results for a function defined on a subset of R can be deduced as a particular case. The results for strong pseudo-concavity, pseudo-concavity and explicit quasi-concavity are analogous and for weakly strong pseudo-convexity (concavity) and weakly convexity (concavity)-like can be established with relevant modifications. It may be remarked here that Arrow and Enthoven [8] and Berge [41] deal with parallel problems for Quasi-convex functions.

Theorem 16. If φ be a function defined on R^n and $\bar{X}$ and $\bar{\bar{X}}$ any pair of points in R^n, and if

$$\psi(u,v) = \varphi(u\bar{X} + v\bar{\bar{X}}) \; ; \; u>0, v>0, u\in R, v\in R \tag{4.3.39}$$

then,

(i) $\varphi \in C^1$ is SPCX(PCX) iff ψ is SPCX(PCX) (4.3.40)

(ii) φ is EQX iff ψ is EQX. (4.3.41)

Proof. Let

$$X_1 = u_1\bar{X} + u_2\bar{\bar{X}} \tag{4.3.42}$$

$$X_2 = u_2\bar{X} + v_2\bar{\bar{X}}$$

and vector $U=(u,v)'$ such that

$$U_1=(u_1,v_1)' \ , \ U_2=(u_2,v_2)'$$

$$\nabla_U \psi(U) = \left(\frac{\partial\psi(U)}{\partial u}, \frac{\partial\psi(U)}{\partial v}\right)'$$

$$\psi(U_1) = \varphi(X_1) \ , \ \psi(U_2) = \varphi(X_2)$$

(i) We have,

$$(U_1-U_2)'\nabla_U\psi(U_2) = (u_1-u_2)\frac{\partial\psi(U_2)}{\partial u_2} + (v_1-v_2)\frac{\partial\psi(U_2)}{\partial v_2}$$

$$= \left((u_1-u_2)\bar{X}' + (v_1-v_2)\bar{\bar{X}}'\right)\nabla_X\varphi(u_2\bar{X}+v_2\bar{\bar{X}})$$

$$= (X_1-X_2)'\nabla_X\varphi(X_2) \qquad (4.3.43)$$

Let φ be SPCX then

$$(X_1-X_2)'\nabla_X\varphi(X_2) \leqslant K(X_1,X_2)\left[\varphi(X_1) - \varphi(X_2)\right] \qquad (4.3.44)$$

(4.3.43) and (4.3.44) yield that

$$(U_1-U_2)'\nabla_U\psi(U_2) \leqslant k(U_1,U_2)\left[\psi(U_1) - \psi(U_2)\right]$$

where $k(U_1,U_2)$ is function of ordered pair (U_1,U_2) and ψ corresponding to $K(X_1, X_2)$. This implies ψ is SPCX. Similarly if we assume ψ be to SPCX, then this assumption, with (4.3.43) gives that φ is SPCX.

We now assume that φ is PCX. Then from (4.3.43) we have,

$$(U_1-U_2)'\nabla_U\psi(U_2) \geqslant 0 \iff (X_1-X_2)'\nabla_X\varphi(X_2) \geqslant 0$$

$$\Rightarrow \varphi(X_1) \geqslant \varphi(X_2)$$

$$\Rightarrow \psi(U_1) \geqslant \psi(U_2)$$

i.e. ψ is PCX.

Similarly assuming that ψ is PCX, we obtain that φ is PCX.

(ii) For $p \in]0,1[$, we have

$$X_p = pX_1 + (1-p)X_2 = (pu_1 + (1-p)u_2)\bar{X} + (pv_1 + (1-p)v_2)\bar{\bar{X}}$$

$$\Rightarrow \varphi(X_p) = \psi[pU_1 + (1-p)U_2] = \psi(U_p) \text{ where } U_p = pU_1 + (1-p)U_2$$

Also $\quad \varphi(X_1) = \psi(U_1)\ , \quad \varphi(X_2) = \psi(U_2)$

Now for $\varphi(X_1) \neq \varphi(X_2)$ we have $\varphi(X_p) < \text{Max}\,[\varphi(X_1), \varphi(X_2)]$

$\Leftrightarrow$ for $\psi(U_1) \neq \psi(U_2)$ we have $\psi(U_p) < \text{Max}\,[\psi(U_1), \psi(U_2)]$

i.e. $\quad \varphi$ is EQX $\Leftrightarrow$ ψ is EQX.

SECTION - IV

CHARACTERIZATION OF SOME PROGRAMMING PROBLEMS:

The main purpose of this section is to characterize, with the help of the result derived in the previous sections of the present chapter, certain problems of mathematical programming as, 'Explicit Quasi-Convex Programming Problems' (EXP.P.'s), 'Pseudo-Convex Programming Problems' (PXP.P.'s) and 'Strong Pseudo-Convex Programming' (SPX-P.P.'s). In general a M.P.P. is stated as

$$\text{Optimize} \quad \varphi(X) \quad \text{for } X \in P, \tag{4.4.1}$$

where, $P \subset R^n$.

If we assume the set P to be a closed convex set, then the problem (4.4.1) is called EX-P.P., PX-P.P. or SPX-P.P. according as the

function φ is EQX, PCX or SPCX. 'Explicit Quasi-Concave Programming Problem, (EV P.P.), 'Pseudo-Concave Programming Problem' (PV P.P.) or 'Strong Pseudo-Concave Programming Problem' (SPV P.P.) is defined analogously.

We have seen in Section I of this chapter that an EQX function, when optimized under suitable assumptions over a convex set, posseses many interesting properties. Furthermore, if the function φ in (4.4.1) be both EQX and EQV, i.e. when φ is EQM then, apart from others, we have the following two interesting properties for the program (4.4.1), under the assumption that P is non-empty and bounded also.

(i) Every local Optimum (maximum or minimum) of φ over P is global optimum (max. or min.) also.

(ii) An optimum occurs at one or more extreme points of the set P.

These two properties always indicate the possibility for developing an 'Extreme Point Procedure' similar to Simplex Method. In, in addition, the set P is such that it contains a finite number of extreme points then the method developed is always finite.

Since the classes of Convex (Concave), Strong Pseudo-Convex (concave) and Pseudo-Convex (concave) an included in the class of Explicit Quasi-Convex (concave) functions therefore functions which are, both convex and concave, Strong Pseudo-Monotonic and Pseudo-Monotic are enriched by the properties of EQM functions.

Below we now classify certain M.P.P.'s according to the nature of their O.F.'s. In each of the following problems, as usual it is assumed that S is a non-empty closed convex subset of R^n. If

necessary, it will be assumed, without specifying, that S is bounded also.

I. EXPLICIT QUASI-CONVEX (CONCAVE) PROGRAMMING

1. NON-LINEAR INDEFINITE FUNCTIONAL PROGRAMMING:

Optimize $\varphi(X) = f(X)\,g(X)$ for $X \in S$,

Where f and g are strictly positive concave functions defined on S and at least one of f and g being non-differentiable over S.

2. NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING

Optimize $\varphi(X) = f(X)/g(X)$ for $X \in S$,

Where, f is non-negative and Convex (concave) and g is strictly positive concave (convex) function with at least one of f and g being non-differentiable over S. If g be linear then non-negativity restriction on f can be removed but then f should be non-differentiable.

3. COMPOSITE FUNCTION PROGRAMMING:

(i) Optimize $\varphi(X) = (g \circ f)(X)$ for $X \in S$,

where f is EQX(EQV) on S and g is a non-decreasing real valued function on an interval $I \subset R$ such that $f(S) \subset I$.

(ii) Optimize $\varphi(X) = g[f_1(X), f_2(X), \ldots, f_p(X)]$ for $X \in S$,

where for i=1,2,------,p, all functions f_i are CX(CV) on the set S with $f_i : S \to I_i \subset R$ and g is a non-decreasing EQX(EQV) function on the set $D = \prod_{i=1}^{p} I_i \subset R^p$, with $g: D \to I \subset R$.

II. PSEUDO-CONVEX (CONCAVE) PROGRAMMING

1. NON-LINEAR INDEFINITE FUNCTIONAL PROGRAMMING:

Optimize $\varphi(X)$ for $X \in S$,

where the function φ is given by

(i) $\varphi = \prod_{j=1}^{p} f_j$

(ii) $\varphi = \prod_{j=1}^{p} (f_j)^{\delta_j}$ where, δ_j for j=1,2,----,p is a non-negative integer,

(iii) $\varphi = \prod_{j=1}^{p} (f_j)^{\frac{1}{\gamma_j}}$ where, for j=1,2,-----,p, (i) γ_j is a positive integer, and (ii) $(f_j)^{\frac{1}{\gamma_j}}$ is strictly positive over S,

(iv) $\varphi = \prod_{j=1}^{p} (f_j)^{m_j/n_j}$ where, (i) m_j, n_j, j=1,2,----,p, are respectively non-negative and positive integers, and (ii) each $(f_j)^{1/n_j}$ is strictly positive over S,

and it being assumed that $f_j \in C^1$, j=1,2,-----,p are strictly positive concave functions on S.

2. NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING:

Optimize $\varphi(X) = \dfrac{f(X)}{g(X)} + \left(\dfrac{f(X)}{g(X)}\right)^3$ for $X \in S$,

where both $f, g \in C^1$ and f is a non-negative CX(CV) function and g is a strictly positive CV(CX) function over S. If g be linear, then non-negativity condition on f is not necessary.

3. COMPOSITE FUNCTION PROGRAMMING:

(i) Optimise $\varphi(X) = (g \circ f)(X)$ for $X \in S$,

where f is PCX(PCV) over the set S and g is a differentiable real valued function with derivative strictly positive on an interval $I \subset R$, such that $f(S) \subset I$.

(ii) Optimise $\varphi(X) = g[f_1(X), f_2(X), \text{----}, f_p(X)]$ for $X \in S$,

where for i=1,2,------,p, all functions $f_i \in C^1$ are CX(CV) on the set S and are such that $f_i: S \rightarrow I_i \subset R$, and $g \in C^1$, is an increasing PCX(PCV) function on the set $D = \prod_{i=1}^{p} I_i \subset R^p$, and is such that $g: D \rightarrow I \subset R$.

III STRONG PSEUDO-CONVEX(CONCAVE) PROGRAMMING

1. NON-LINEAR FRACTIONAL FUNCTIONSL PROGRAMMING:

Optimise $\varphi(X) = \dfrac{f(X)}{g(X)}$ for $X \in S$,

where, $f \geqslant 0$, is CX(CV) on S, $g > 0$ is CV(CX) on S and both $f, g \in C^1$ on S. If g is linear then condition of non-negativity is not necessary on f.

2. COMPOSITE FUNCTION PROGRAMMING:

Optimise $\varphi(X) = g[f_1(X), f_2(X), \text{----}, f_p(X)]$ for $X \in S$,

where for i=1,2,------,p, all function $f_i \in C^1$ are CX(CV) on the set S and are such that $f_i: S \rightarrow I_i \subset R$, and g is an increasing SPCX(SPCV) function on the set $D = \prod_{i=1}^{p} I_i \subset R^p$ and is such that

$g: D \rightarrow I \subset R$.

IV. STRONG PSEUDO-MONOTONIC PROGRAMMING:

A L.F.F.P. problem

Optimize $\varphi(X) = \dfrac{c'X+\alpha}{d'X+\beta}$

subject to

$$A X \leq b$$
$$X \geq 0$$

where symbols have their usual meanings and the set $S=\{x;\ Ax\leq b,\ x\geq 0\}$ being assumed to be non-empty compact convex set, can be characterized as a strong pseudo-monotonic programming problem.

SECTION - V

PROGRAMMING PROBLEMS WITH CONVEX FRACTIONAL FUNCTIONS:

The main purpose of this section is to characterize certain mathematical programming problems as convex programming problems. If f be a CX and non-negative function and g be a CV and strictly positive function over a convex set $S \subseteq R^n$, then with the help of results established in the previous sections of this chapter. We can only assert that f^2/g is EQX (and also QX) over S, and if both $f, g \in C^1$ also, then f^2/g can be asserted to be SPCX over S. The main result proved in this section is that in such a case, f^2/g is always a CX function over S, and it is believed that this result will be of great advantage in N.L.F.F.P.P.'s. A few applications of this result to M.P. have been considered.

MAIN THEOREM: We now prove the following main theorem of this section.

"If f be a CX and a non-negative function and g be a CV and strictly positive function over a convex set $S \subseteq R^n$, then f^2/g is CX over S."

Proof. Let for all $\lambda \in [0,1]$, and every X_1, X_2 in S,

$$E = \lambda \frac{[f(x_1)]^2}{g(x_1)} + (1-\lambda)\frac{[f(x_2)]^2}{g(x_2)} - \frac{[f(\lambda x_1 + \overline{1-\lambda}\, x_2)]^2}{g(\lambda x_1 + \overline{1-\lambda}\, x_2)}$$

Using the convexity and non-negativity of f, and concavity and strict positivity of g, we get for all $\lambda \in [0,1]$,

$$E \geqslant \lambda \frac{[f(x_1)]^2}{g(x_1)} + (1-\lambda)\frac{[f(x_2)]^2}{g(x_2)} - \frac{[\lambda f(x_1) + (1-\lambda) f(x_2)]^2}{\lambda g(x_1) + (1-\lambda) g(x_2)}$$

$$= \frac{\lambda(1-\lambda)\left\{[f(x_1)]^2[g(x_2)]^2 + [f(x_2)]^2[g(x_1)]^2 - 2 f(x_1) g(x_1) f(x_2) g(x_2)\right\}}{g(x_1)\, g(x_2)\,[\lambda g(x_1) + (1-\lambda) g(x_2)]}$$

$$= \frac{\lambda(1-\lambda) g(x_1) g(x_2)}{\lambda g(x_1) + (1-\lambda) g(x_2)} \left[\frac{f(x_1)}{g(x_1)} - \frac{f(x_2)}{g(x_2)}\right]^2$$

$$\geqslant 0$$

This proves that f^2/g is CX over S.

Remark 1. In the above theorem if we assume that both the functions f and g are linear, then the condition of non-negativity need not be placed upon the function f.

Remark 2. We can now prove easily the following more results given in the Table-1 below.

TABLE - 1

f	SX	CX	SX	CV	SV	CV	SV	CV	CV	SV	SV	CX	SX	CX	SX
	>0	≥ 0	>0	≤ 0	<0	≤ 0	<0	≤ 0	≤ 0	<0	<0	≥ 0	>0	≥ 0	>0
g	CV	SV	SV	CV	CV	SV	SV	CX	SX	CX	SX	CX	CX	SX	SX
	>0	>0	>0	>0	>0	>0	>0	<0	<0	<0	<0	<0	<0	<0	<0
f^2/g	SX	SX	SX	CX	SX	SX	SX	CV	SV	SV	SV	CV	SV	SV	SV

Below now we state some interesting result in the form of corollaries, which can be easily proved.

Corollary 1. If g be a CV and strictly positive function over a convex set $S \subset R^n$, then $\frac{1}{g}$ is always CX over S.

Remark 3. Corollary 1 contains a well known results used, e.g. by Anthony V. Fiacco and Garth P. McCormick[1] [78] . It may also be observed here that if the function f be CX and strictly positive over a convex set $S \subset R^n$, then $\frac{1}{f}$ is not necessarily CV, e.g. let

$$f(x) = e^{x} \qquad \text{for } x \in R$$

then

$$\frac{1}{f(x)} = e^{-x} \qquad \text{for } x \in R$$

and we know that both e^{x} and e^{-x} for $x \in R$ are CX.

Corollary 2. If f be a CX and non-negative (or CV and non-positive function and g be a CV and strictly positive function on a convex set $S \subset R^n$, then

1. After the author had deduced this corollary 1, the reference [78] was brought to his notice by one of the referees of [21] .

$$\frac{l_0 + l_1\psi_1(x) + l_2\psi_2(x) + \text{-----} + l_p\psi_p(x)}{g(x)}$$

is also CX over S, provided for K=1,2,-----,p (p being finite) (i) $\psi_k(X) = [f(X)]^{2^k}$ for $X \in S$, and (ii) l_k are known non-negative real numbers.

Corollary 3. If over a convex set $S \subset R^n$, f be non-negative and CV, g be strictly positive and CV and h be strictly positive and CX, then $\frac{fg}{h^2}$ is EQV and if, in addition, all f,g and $h \in C^1$ then $\frac{fg}{h^2}$ is SPCV. Furthermore, if f,g and h be such that $f>0, g>0$ and $h \geqslant 0$ then $\frac{h^2}{fg}$ is EQX (SPCX when f,g, $h \in C^1$), and in this case if all f,g and h are linear functions on S, then we need not restrict h to be non-negative whereas $\frac{h^2}{fg}$ will still remain SPCX on S.

Remark 4. From Corollary 3, it is easily deduced that if both $f,g \in C^1$ and are respectively non-negative and strictly positive CV function on a convex set $S \subset R^n$, then N.L.I.F.P.P., Optimize $\varphi(X) = f(X)g(X)$ for $X \in S$, can be classified as 'Strong Pseudo-Concave Programming' problem. It is to remark over here that now it has been possible for us to relax the original condition of strict positivity on f to non-negativity. Also it is to note that if at least one of f,g, $h \notin C^1$ then the programming problem, optimise $\varphi(X) = \frac{f(X)g(X)}{h^2(X)}$ is in the calss of EV P.P.'s and if all f,g,$h \in C^1$ then it belongs to Strong Pseudo-Concave Programming. Evidently it is seen that Indefinite Quadratic Programming Problem considered by Kanti Swarup [124,125,126] is a SPV P.P.

Corollary 4. If g_1, g_2 be non-negative CX functions f_1, f_2 and f_3 be strictly positive CV functions, on a convex set $S \subset R^n$, then

$\frac{g_1^2}{f_1 f_2} + \frac{g_2^2}{f_1 f_3}$ is EQX or SPCX according as at least one of f_1, f_2, f_3, g_1, g_2, $\notin C^1$, for all of f_1, f_2, f_3, g_1, g_2, $\in C^1$.

Applications. Below we discuss some applications of the results deduced above to mathematical programming problems.

Note. Throughout the section, the set S will mean a non-empty closed convex subset of R^n and if necessary assumed to be bounded also without making any such specifications. Also the P will mean the closed convex polyhedral set given by $P = \{X ;\ AX \leq b,\ X \geq 0,\ X \in R^n\}$ assumed to be regular (i.e. bounded and non-empty).

Problem 1. We consider the fractional functional programming problem,

$$\text{Optimize} \quad \varphi(X) = c'X + \alpha + \frac{X'HX + \alpha_1}{d'X + \beta_1} + \frac{X'GX + \alpha_2}{p'X + \beta_2} \qquad (4.5.1)$$

over the set S, where,

(i) c,d and p are nx1 column vectors,

(ii) H and G are real symmetric positive semi-definite matrices of order nxn.

(iii) α, α_1, α_2, β_1 and β_2 are arbitrary scalar constants, and

(iv) $d'X + \beta_1$, $p'X + \beta_2$ are strictly positive over S.

Since $(X'HX)^{1/2}$ and $(X'GX)^{1/2}$ are CX [70] , and $\frac{\alpha_1}{d'X+\beta_1}$, $\frac{\alpha_2}{p'X+\beta_2}$ are CX by Corollary 1, therefore $\frac{X'HX+\alpha_1}{d'X+\beta_1}$ and $\frac{X'GX+\alpha_2}{p'X+\beta_2}$ are CX. Hence the above problem is a Convex Programming Problem.

Problem 2. We consider the M.P.P.,

$$\text{Optimize} \quad \varphi(X) = \frac{(p'X+\gamma)^2}{d'X+\beta} \tag{5.4.2}$$

over the set S, where,

(i) p' is an nx1 column vector,

(ii) β and γ are arbitrary scalar constants.

(iii) $d'X+\beta > 0$ over the set S.

It is easily seen with the help of 'Main Theorem' and 'Remark 1' that the Problem 2 above is a convex programming problem.

Problem 3. Indefinite Quadratic Fractional Functional Programming

The problem considered here is as follows.

$$\text{Minimize} \quad \varphi(X) = \frac{(p'X+\gamma)^2}{(c'X+\alpha)(d'X+\beta)} \quad \text{for} \quad X \in P, \tag{5.4.3}$$

where,

(i) c,d,p, α, β and γ and the set P are as already explained.

(ii) $c'X+\alpha$, $d'X+\beta$ are strictly positive over P.

It is easily seen, with the help of Corollary 3 above, that the problem (5.4.3) is SPX P.P. It is shown here that such a 'Non-Convex Programming Problem' can be solved by solving a convex fractional functional programming problem of the type given in Problem 2. For that we introduce the transformation of variables

$$Y = y_0 X \tag{5.4.4}$$

which was originally employed by Charnes and Cooper [45] and is a homeomorphism with the scalar $y_0 \geq 0$ to be so chosen that

$$d'Y + \beta y_0 = v \tag{5.4.5}$$

where $\upsilon > 0$ is a specified number.

With the help of (5.4.4) and (5.4.5) the problem (5.4.3) reduces to

$$\text{Minimize} \quad \psi(y_0, Y) = \frac{(p'Y + \gamma y_0)^2}{c'Y + \alpha y_0} \tag{5.4.6-a}$$

subject to

$$\left.\begin{aligned} AY - by_0 &\leq 0 \\ d'Y + \beta y_0 &= \upsilon \\ y_0, Y &\geq 0 \end{aligned}\right\} \tag{5.4.6-b}$$

Below, we now state a result, to be used later on, in the form of a lemma, which can be found in the reference quoted.

Lemma 1. Every (y_0, Y) satisfying (5.4.6b) has $y_0 > 0$ [45].

Theorem 1. If (y_0^* , Y^*) is an optimal solution to the problem (5.4.6), then $\left(Y^*/y_0^*\right)$ is an optimal solution to the problem (5.4.3).

Proof. If possible let the above assertion be false and that there exists an optimal solution $X^* \in P$ such that,

$$\frac{(p'X^* + \gamma)^2}{(c'X^* + \alpha)(d'X^* + \beta)} > \frac{\left[p'\left(Y^*/y_0^*\right) + \gamma\right]^2}{\left[c'\left(Y^*/y_0^*\right) + \alpha\right]\left[d'\left(Y^*/y_0^*\right) + \beta\right]} \tag{5.4.7}$$

Now as $d'X^* + \beta$ is strictly positive, therefore,

$$d'X^* + \beta = \theta\upsilon \qquad \text{for some} \quad \theta > 0$$

We now consider,

$$\hat{Y} = \frac{X^*}{\theta}, \quad \hat{y}_0 = \frac{1}{\theta}$$

Then $\frac{1}{\theta}\left[d' X^* + \beta\right] = d\hat{Y} + \beta\hat{y}_0 = v$ (5.4.8)

and $\frac{1}{\theta}\left[A X^* - b\right] \leqslant 0$

i.e. $A\hat{Y} - b\hat{y}_0 \leqslant 0$ (5.4.9)

From (5.4.8) and (5.4.9) we infer that $(\hat{y}_0, \hat{Y})$ is a feasible solution to (5.4.6).

Taking L.H.S. of (5.4.7) we have,

$$\frac{(p' X^* + \gamma)^2}{(c' X^* + \alpha)(d' X^* + \beta)} = \frac{\left[p'(X^*/\theta) + \gamma/\theta\right]^2}{\left[c'(X^*/\theta) + \alpha/\theta\right]\left[d'(X^*/\theta) + \beta/\theta\right]}$$

$$= \frac{(p'\hat{Y} + \gamma\hat{y}_0)^2}{(c'\hat{Y} + \alpha\hat{y}_0)(d'\hat{Y} + \beta\hat{y}_0)}$$

$$= \frac{(p'\hat{Y} + \gamma\hat{y}_0)^2}{(c'\hat{Y} + \alpha\hat{y}_0)v} \tag{5.4.10}$$

Taking R.H.S. of (5.4.7) we obtain

$$\frac{\left[p'(Y^*/y_0^*) + \gamma\right]^2}{\left[c'(Y^*/y_0^*) + \alpha\right]\left[d'(Y^*/y_0^*) + \beta\right]} = \frac{(p' Y^* + \gamma y_0^*)^2}{(c' Y^* + \alpha y_0^*)v} \tag{5.4.11}$$

Using (5.4.10) and (5.4.11) in (5.4.7) we have,

$$\frac{(p'\hat{Y} + \gamma\hat{y}_0)^2}{(c'\hat{Y} + \alpha\hat{y}_0)v} > \frac{(p' Y^* + \gamma y_0^*)^2}{(c' Y^* + \alpha y_0^*)v}$$

$$\Rightarrow \quad \frac{(p'\hat{Y} + \gamma\hat{y}_0)^2}{c'\hat{Y} + \alpha\hat{y}_0} > \frac{(p' Y^* + \gamma y_0^*)^2}{c' Y^* + \alpha y_0^*} \tag{5.4.12}$$

In equation (5.4.12) implies that ($\hat{y}_0, \hat{Y}$) and not (y_0^*, Y^*) is an optimal solution to the problem (5.4.6), which is a contradiction.

Thus we see that for any regular set P given by $P=\{X; AX \leq b, X \geq 0\}$ to solve the given problem (5.4.3) it suffices to solve the problem,

$$\text{Minimize} \quad \psi(y_0, Y) = \frac{(p'Y + \gamma y_0)^2}{c'Y + \alpha y_0} \tag{5.4.13a}$$

subject to

$$\left.\begin{aligned} AY - by_0 &\leq 0 \\ d'Y + \beta y_0 &= 1 \\ y_0, Y &\geq 0 \end{aligned}\right\} \tag{5.4.13b}$$

Problem (5.4.13) is easily seen to be a convex programming problem. Thus we see that the Strong Pseudo-Convex Programming (Non-Convex Programming) Problem (5.4.3) can be solved by solving the convex programming problem (5.4.13).

Problem 4. Here we consider the programming problem,

$$\text{Optimise} \quad \varphi(X) = \frac{X'HX + (X'GX)^{1/2} + c'X + \alpha}{d'X + \beta} \tag{5.4.14}$$

for $X \in P$, where,

(i) The symbols c, d, α and β are as usual.

(ii) G and H are nxn order real symmetric positive semi-definite matrices.

(iii) $d'X + \beta > 0$ over the set P.

Problem (5.4.14) is easily seen to be that of EX P.P. and if G

is assumed to be a null matrix, then it is a problem of SPX P.P. (However, Aggarwal [1] , with G as null matrix, proves the O.P. to be EQX). If we take the matrix H to be null then the problem (5.4.14) reduces to the type considered by Aggarwal [3] , and both the matrices G and H be assumed to be null matrices, the problem (5.4.14) reduces to the well known problem of L.F.F.P.

By employing the transformation (5.4.4) it is now easily seen that the problem (5.4.14) (with relevant modifications for particular cases [1, 3]) can be solved by obtaining the solution of the following convex programming problem,

$$\text{Optimize} \quad \Psi(y_0, Y) = \frac{Y'HY}{y_0} + (Y'GY)^{1/2} + c'Y + \alpha y_0 \tag{5.4.15a}$$

subject to

$$\begin{aligned} AY - by_0 &\leq 0 \\ dY + \beta y_0 &= 1 \\ y_0, Y &\geq 0 \end{aligned} \tag{5.4.15b}$$

Problem 5. Here we consider the problem of "QUADRATIC FRACTIONAL FUNCTIONAL PROGRAMMING WITH IRRATIONAL FUNCTIONS IN THE OBJECTIVES" which is a slight generalisation of problem (5.4.14) and special cases of which were considered as particular cases to N.L.F.F.P. in Chapter III. Problem considered here is stated as:

$$\text{Optimize} \quad \varphi(X) = \frac{X'HX + (X'GX)^{1/2} + c'X + \alpha}{X'EX + (X'FX)^{1/2} + d'X + \beta} \tag{5.4.16}$$

for $X \in P$, where,

(i) E and F are nxn order real symmetric positive semi-definite matrices and other symbols are as in problem (5.4.14).

(ii) $X'HX + (X'GX)^{1/2} + c'X + \alpha \geq 0$, $X'EX + (X'FX)^{1/2} + d'X + \beta > 0$, over the set P and when the matrices E and F are null matrices, $X'HX + (X'GX)^{1/2} + c'X + \alpha$ is unrestricted in sign.

Using the transformation (5.4.4), problem (5.4.16) reduces to,

$$\text{Optimize} \quad \Psi(y_0, Y) = \frac{Y'HY}{y_0} + (Y'GY)^{1/2} + c'Y + \alpha y_0 \tag{5.4.17a}$$

subject to

$$\left.\begin{aligned} AY - by_0 &\leq 0 \\ \frac{Y'EY}{y_0} + (Y'FY)^{1/2} + d'Y + \beta y_0 &= 1 \\ y_0, Y &\geq 0 \end{aligned}\right\} \tag{5.4.17b}$$

in which, although the O.F. and all the constraint functions are convex yet the problem is not that of convex programming because of the nature of the non-linear constraint. If we set the matrices G and F as null matrices, then we get the problem of Programming with Quadratic Fractional Functionals", as considered by Kanti Swarup [121]. Such a problem is stated as

$$\text{Optimize} \quad \varphi(X) = \frac{X'HX + c'X + \alpha}{X'EX + d'X + \beta} \tag{5.4.18}$$

subject to the vector $X \in P$, where

(i) $X'HX + c'X + \alpha \geq 0$, $X'EX + d'X + \beta > 0$, (ii) and when the matrix E is a null matrix, $X'HX + c'X + \alpha$ is unrestricted in sign.

Kanti Swarup [121] reduced the problem (5.4.18) to two quadratic programming problems in which all other constraints are linear and one is quadratic but from which the case of L.F.P.P. is not deducible as a particular case. Here we discuss the problem (5.4.18) in three cases as follows.

Case 1. Other symbols remaining unchanged, when the matrices E and H are respectively, negative semi-definite and positive semi-definite matrices, each of order nxn, problem (5.4.18) belongs to the class of SPX P.P.'s.

Case 2. Other symbols being same, when the matrices E and H are respectively, real symmetric positive semi-definite and negative semi-definite, each of order nxn then the problem (5.4.18) is a SPV P.P.

Case 3. When both the matrices E and H are nxn order positive semi-definite then since we do not know the nature of the O.F. φ in this case, therefore, we reduce it to

$$\text{Optimize} \quad \psi(y_0, Y) = \frac{Y'HY}{y_0} + c'Y + \alpha y_0 \tag{5.4.19a}$$

subject to

$$\left.\begin{array}{r} AY - by_0 \leq 0 \\ \dfrac{Y'EY}{y_0} + d'Y + \beta y_0 = 1 \\ y_0, Y \geqslant 0 \end{array}\right\} \tag{5.4.19b}$$

which again is a non-convex programming problem because of the reasons, obvious, but we see that when the matrices E and H become null matrices, then the case for L.F.P., similar to considered by Charnes and Cooper [45], follows as a particular case.

Problem 6. "Indefinite Cubic Programming with Standard Errors in the Objective Function." We consider here the problem considered by Bector [25], which is stated as,

$$\text{Maximize} \quad \varphi(X) = \left[c'X - X'EX - (X'GX)^{1/2} + \alpha\right]\left[d'X + \beta\right] \tag{5.4.20}$$

for $X \in P$, where

(i) E and G are real symmetric positive semi-definite matrices of order nxn, and

(ii) the remaining symbols have unchanged meanings, $c'X - X'EX - (X'GX)^{1/2} + \alpha > 0$ and $d'X + \beta > 0$ over P.

The problems (5.4.20) is that of EV P.P. (SPV P.P. if G be a null matrix). Here we show that it can be solved by solving another convex programming problem.

Employing the transformation (5.4.4) the problem (5.4.20) reduces to,

$$\text{Maximise } \psi(y_0, Y) = \frac{c'Y - \dfrac{Y'EY}{y_0} - (Y'GY)^{1/2} + \alpha y_0}{y_0^2}$$

subject to

$$AY - by_0 \leq 0$$

$$d'Y + \beta y_0 = 1$$

$$y_0, Y \geq 0$$

But we know that,

$$\text{Max } \frac{c'Y - \dfrac{Y'EY}{y_0} - (Y'GY)^{1/2} + \alpha y_0}{y_0^2} = \frac{1}{\text{Min } \dfrac{y_0^2}{c'Y - \dfrac{Y'EY}{y_0} - (Y'GY)^{1/2} + \alpha y_0}}$$

Therefore, the non-convex programming problem (5.4.20) can be solved by solving the problem

$$\text{Minimise } \eta(y_0, Y) = \frac{y_0^2}{c'Y - \dfrac{Y'EY}{y_0} - (Y'GY)^{1/2} + \alpha y_0} \qquad (5.4.21)$$

subject to

$$\left.\begin{aligned} AY - by_0 &\leq 0 \\ d'Y + \beta y_0 &= 1 \\ y_0, Y &\geq 0 \end{aligned}\right\} \tag{5.4.21b}$$

which is a convex programming problem.

If we set the matrix E to be null, then the problem reduces to that considered by Bector in [21], and if we take both the matrices E and G to be null matrices, the problem (5.4.20) reduces to the problem considered by Kanti Swarup [124].

Problem 7. Slightly more general mathematical models of the problem (5.4.20) could be stated as

(i) $$\underset{X \in P}{\text{Maximize}}\ \varphi(X) = \left[c'X - X'EX - (X'GX)^{1/2} + \alpha\right]\left[d'X - X'FX - (X'HX)^{1/2} + \beta\right] \tag{5.4.22}$$

(ii) $$\underset{X \in P}{\text{Maximize}}\ \varphi(X) = \left[c'X - X'EX - (X'GX)^{1/2} + \alpha\right]\left[d'X + X'FX + (X'HX)^{1/2} + \beta\right] \tag{5.4.23}$$

where all the matrices E, F, G and H are real symmetric positive semi-definite with other symbols as usual, and each of the factors in the O.F. for both the problems is strictly positive over P.

First problem is obviously EV P.P. (when matrices G and H are null. it is SPV P.P.) whereas the nature of the O.F. in the second problem is unknown. Such a problem can be reduced to another problem in which the O.F. is same as in (5.4.21a) and all the constraints are convex but the solution set (set of feasible solution) given by

$$\begin{aligned} AY - by_0 &\leq 0 \\ d'Y + \frac{Y'FY}{y_0} + (Y'HY)^{1/2} + \beta y_0 &= 1 \\ y_0, Y &\geq 0 \end{aligned}$$

is non-convex because of the obvious reasons.

Problem 8. Jagannathan [116] considered the problem

$$\text{Minimize } \varphi(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} \frac{c_j}{x_j + s_j}$$

in which $c_j > 0$ and $s_j > 0$; subject to $Ax \leq b,\ x \geq 0$ where A, b have the usual meanings. The above problem arose in study of problems of general allocation in 'multiple character studies'.

With the help of the main theorem and the fact that the sum of two CX functions is also CX, we see that the above problem is that of convex programming. This result is in conformity with that deduced by Jagannathan [116] .

CHAPTER - V

METHOD OF FEASIBLE DIRECTION FOR STRONG PSEUDO-CONCAVE (PSEUDO-CONCAVE) PROGRAMMING; A SPECIAL TYPE OF NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING

INTRODUCTION:

The purpose of this chapter is to develop computational techniques for certain mathematical programming problems. The chapter is mainly divided into two sections. Section I is devoted to the development of a computational technique, similar to 'Method of Feasible Directions' [184,185] , for a M.P.P. in which the O.F. is SPCV and the constraint set is constrained by non-linear differential quasi-convex functions and properly specified linear and non-negativity restrictions, under regularity condition [11] . The essential difference between the problem considered by Zoutendijk [184,185] and the problem considered

here lies in the fact that in the present problem the O.F. to be maximized is SPCV and the non-linear constraint functions are CX, where as in the problem considered by Zoutendijk the O.F. is CV and the non-linear constraint functions are CX. Since the SPCV functions posses the property of having every local maximum as global maximum, therefore we hope that the local maximum obtained with the help of method of feasible directions, will be a global maximum. Furthermore, we have extended the method for the problem in which the O.F. is PCV, mentioning explicitly the difference of assumptions between the two problems.

Method of feasible directions is advantageous in the sense that, firstly to initiate the computations we can use any feasible solution at which the gradient of the O.F. does not vanish and it is not necessary that we should have some special form of a feasible solution (such as a B.F.S.), and secondly we may proceed into the interior of the feasible domain to increase or decrease the O.F. which usually provides a faster convergence. Section II deals with the optimization of a special type of non-linear fractional functional over a closed convex polyhedra set. The O.F. is proved to be PCM over the constraint set. This makes it possible to develop a finite iteration technique to obtain the global optimum of the O.F. Method develop is similar to that of Beale [16], as developed by Kanti Swarup [122] for L.F.F.P. It is interesting to remark here that the property of global optimum having been taken on at one or more of the extreme points of the constraint set follows as a consequence of the method developed also.

SECTION - I

METHOD OF FEASIBLE DIRECTIONS FOR STRONG PSEUDO-CONCAVE PROGRAMMING

(PSEUDO-CONCAVE PROGRAMMING)

The problem considered here is stated as

$$\text{Maximize } \varphi(X) \qquad \text{for } X \in P, \tag{5.1.1}$$

where,

(i) the constraint set $P \subset R^n$ is given by

$$P = \left\{ X \,;\, g_i(X) \leq b_i,\ i \in N = 1, 2, \ldots, v;\ a_i X \leq b_i,\ i \in L = v+1, \ldots, m;\ 0 \leq X \leq p \right\} \tag{5.1.2}$$

it being given that all the functions $g_i(X) \in C^1$ for $i \in N$ and are QX over R^n; a_i, $i \in L$ are 1xn row vectors; b_i, $i \in N \cup L$ are given scalars, p is an nx1 column vector, such that the set P in (5.1.2) is a closed convex set assumed to be non-empty.

(ii) The function $\varphi \in C^1$ is SPCV(PCV).

Before providing the outlines of the actual algorithm, we give, below, certain pre-requisites in the form of notations and definitions etc. and develop some theory necessary for the development of the algorithm.

PRE-REQUISITES:

Throughout this section we shall follow the following notations and terminology.

(i) $\eta(X)$, an nx1 vector stands for $\nabla_x \varphi(X)$ such that $\eta'(X)$ is the transpose of $\eta(X)$. Similarly $\gamma_i(X) = \nabla_x g_i(X)$, $i \in N$ and $\gamma_i'(X)$ for $i \in N$ is the transpose of $\gamma_i(X)$.

(ii) A FEASIBLE DIRECTION [185, page 21] : Let $X_k \in P$. A direction d_k through X_k is called feasible if we do not immediately leave the

region P while making a sufficiently small move in the direction d_k. Hence, if there exists a scalar $\lambda_k > 0$ such that $X_k + \lambda_k d_k \in P$ holds, where d_k is an nx1 column vector, then d_k is a feasible direction. (5.1.3)

(iii) A USABLE FEASIBLE DIRECTION (u.f.d.) [185, page 22]. A feasible direction d_k through $X_k \in P$ is called a u.f.d. if it satisfies,

$$\left[\frac{d\,\varphi(x_k+\lambda d_k)}{d\lambda}\right]_{\lambda=0} = \eta'(x_k)\,d_k > 0 \qquad (5.1.4)$$

(iv) NORMALISATION REQUIREMENT[1] [185, page 70] : A u.f.d. d_k is said to satisfy a normalisation requirement if it satisfies one of the following.

$N(1).\quad d_k' d_k = 1\ ;\ N(2).\ -1 \le d_{kj} \le 1$ for all $j = 1, 2, \ldots, n$ (5.1.5)

(v) N_a^k, L_a^k, J, J_a^{k+}, J_a^{k-}, $D(x_k)$ and $D(x_k, \sigma)$ shall denote respectively,

$$N_a^k = \left\{ i\ ;\ g_i(x_k) = b_i\ ,\ i \in N \right\} \qquad (5.1.6)$$

$$L_a^k = \left\{ i\ ;\ a_i x_k = b_i\ ,\ i \in L \right\} \qquad (5.1.7)$$

$$J = (1, 2, \ldots, n) \qquad (5.1.8)$$

$$J_a^{k+} = \left\{ j\ ;\ x_{kj} = 0\ ,\ j \in J \right\} \qquad (5.1.9)$$

$$J_a^{k-} = \left\{ j\ ;\ x_{kj} = p_j\ ,\ j \in J \right\} \qquad (5.1.10)$$

$$D(x_k) = \left\{ d_k\ ;\ g_i'(x_k)\,d_k \le 0\ ,\ i \in N_a^k\ ;\ a_i d_k \le 0\ ,\ i \in L_a^k\ ,\ \begin{matrix} d_{kj} \ge 0\ ,\ j \in J_a^{k+} \\ d_{kj} \le 0\ ,\ j \in J_a^{k-} \end{matrix} \right\} \qquad (5.1.11)$$

1. Zoutendijk [185, page 70] has given three more Normalisations.

$$D(X_k,\sigma)=\left\{\begin{array}{l}(d_k,\sigma);\ g'_i(X_k)d_k+\theta_i\sigma\leq 0,\ i\in N_a^k,\ a_i d_k\leq 0,\ i\in L_a^k, \\ \qquad d_{kj}\geq 0,\ j\in J_a^{k+},\ d_{kj}\leq 0,\ j\in J_a^{k-}\end{array}\right\}$$

where σ is said to be an extra variable and θ_i is a positive number, choice for which may be arbitrary [185] . (5.1.12)

(vi) A BEST USABLE FEASIBLE DIRECTION (b.u.f.d.) [185] :

(a) When the set N in (5.1.2) is non-null, we call the vector d_k to be b.u.f.d. (or sometimes 'BEST DIRECTION') through $X_k \in P$, if (d_k,σ) satisfies the following requirements.

1. $(d_k,\sigma)\in D(X_k,\sigma)$.

2. $-\eta'(X_k)d_k+\sigma\leq 0$. (5.1.13)

3. Normalization requirement (5.1.5).

4. σ to be maximized .

(b) When the set N in (5.1.2) is null, we call the vector d_k to be the b.u.f.d. through $X_k \in P$, if it satisfies the following requirements.

1. $d_k \in D(X_k)$.

2. Normalisation requirement (5.1.5). (5. .14)

3. $\eta'(X_k)\,d_k$ is maximized .

REGULARITY CONDITION (C_1): Given an $X_k \in P$ such that all constraint functions belong to the class C^1 at X_k . We say the point X_k satisfies the 'Regularity Condition' [11] for the system

$$\left.\begin{array}{lll} g_i(X)\leq b_i & & i\in N \\ a_i X\leq b_i & & i\in L\end{array}\right\} \qquad (5.1.15)$$

if there exists some vector d_k satisfying

$$\left.\begin{array}{ll} \varphi_i'(X_k)\, d_k < 0 & i \in N_a^k \\ a_i d_k \leq 0 & i \in L_a^k \end{array}\right\} \qquad (5.1.16)$$

For the programming problem we now make the following assumptions.

1 (a) When the O.F. φ is SPCV. Let m be the maximum of φ on P.
We assume that the SPCV function φ is either unbounded on P ($m = \infty$) or that

$$\left.\begin{array}{l} P_m = \{X\,;\, X \in P,\ \varphi(X) = m\} \text{ is bounded.} \\ \text{and } K(X_1, X_2) < \infty \quad \forall\, X_1, X_2 \in P. \end{array}\right\} \qquad (5.1.17)$$

1 (b) When the O.F. φ is PCV: Either φ is unbounded on the set
P or the set,

$$P_\alpha = \{X\,;\, \varphi(X) \geqslant \alpha,\ X \in P\} \qquad (5.1.18)$$

is bounded for all $\alpha \in R$.

2. Every point $X \in P$, satisfies the regularity condition (C_1).
We now develop the prerequisited theor for the development of the computational algorithm.

Lemma 1. A function $f(X)$ for $X \in R^n$ is SPCV(SPCX) or PCV(PCX) iff one dimensional function $\psi(\lambda) = f(X + \lambda d)$ is SPCV (SPCX) or PCV(PCX) for any two vectors X and d and for all $\lambda > 0$.

Proof of the lemma follows as a particular case of Theorem 16 of Section III of Chapter IV.

Lemma 2. Let $X_k \in P$ be any known feasible solution and d_k a known direction through X_k. If $X(\lambda) = X_k + \lambda d_k$, for however small $\lambda > 0$ be any point on a ray emanating from X_k along d_k and $\psi(\lambda) = \varphi[X(\lambda)]$, then at X_k, $\psi(\lambda)$ is an increasing function of λ

along d_k iff d_k is a u.f.d. at X_k.

Proof. We know that,

$$\text{Rate of change of } \psi(\lambda) \text{ for } \lambda=0 \text{ along } d_k = \left[\frac{d\,\psi(\lambda)}{d\lambda}\right]_{\lambda=0}$$
$$= \left[\frac{d\,\varphi(x_k+\lambda d_k)}{d\lambda}\right]_{\lambda=0}$$
$$= \eta'(x_k)\,d_k$$

Let us first assume d_k to be u.f.d. through X_k. Therefore, from (5.1.4) and (5.1.19) we obtain that,

Rate of change of $\psi(\lambda)$ at $\lambda=0$ along d_k is >0 i.e. at X_k the function $\psi(\lambda)$ is an increasing function of λ along d_k in a neighbourhood of $\lambda=0$. Again, assume that at X_k, the function $\psi(\lambda)$ is an increasing function of λ in the direction d_k through X_k. Therefore, at $\lambda=0$ the rate of change of $\psi(\lambda)$ in the direction d_k is >0. This implies from (5.1.19) that $\eta'(x_k)d_k>0$ i.e. d_k is a u.f.d. through X_k.

Remark 1. In Lemma 2 we have not assumed the strong pseudo-concavity (pseudo-concavity) character on φ. Below, with the help of Lemmas 1 and 2, we now prove a very interesting property of SPCV(PCV) functions.

Theorem 1. If, in the program (5.1.1), d_k be a u.f.d. through the point $x_k \in P$, then at X_k the function φ increases along d_k and goes on increasing till it takes on its first global maximum in that direction.

Proof. Let $\psi(\lambda)=\varphi(x_k+\lambda d_k)$ for however small $\lambda>0$. Now d_k is a u.f.d. through X_k, therefore, by Lemma 2, at $\lambda=0$ (i.e X_k) there is a small neighbourhood $\mathcal{N}_\lambda$ along the direction d_k, such that in this neighbourhood $\psi(\lambda)$ is an increasing function of λ

Since all g_i for $i \in N$ are QX, therefore (5.1.21) implies that

$$\lambda d_k' q_i(x_k) \leq 0 \qquad \text{for} \quad i \in N_a^k$$

$$\text{i.e.} \quad q_i'(x_k)\, d_k \leq 0 \qquad \text{for} \quad i \in N_a^k \tag{5.1.22}$$

Also $\qquad a_i(x_k + \lambda d_k) \leq a_i x_k \quad$ for $\quad i \in L_a^k$

This implies

$$a_i\, d_k \leq 0 \qquad \text{for} \quad i \in L_a^k \tag{5.1.23}$$

Similarly we obtain

$$d_{kj} \geq 0 \qquad \text{for} \quad j \in J_a^{k+} \tag{5.1.24}$$

$$d_{kj} \leq 0 \qquad \text{for} \quad j \in J_a^{k-} \tag{5.1.25}$$

Conditions given by (5.1.22) to (5.1.25) imply that $d_k \in D(x_k)$.

Theorem 4. If $d_k \in D(x_k)$ with $q_i'(x_k)\, d_k < 0$, then d_k is a feasible direction through $x_k \in P$.

Proof. It is given that $d_k \in D(x_k)$ with $q_i'(x_k)\, d_k < 0$. This implies that if we move from x_k along d_k then in a δ-neighbourhood $\mathcal{N}_\delta(x_k)$ of x_k, $g_i(x)$ decreases, i.e. there exists a point $x_k + \lambda d_k$ for however small $\lambda > 0$, such that

$$g_i(x_k + \lambda d_k) < g_i(x_k) = b_i \quad \text{for } i \in N_a^k \text{ and } x_k + \lambda d_k \in \mathcal{N}_\delta(x_k).$$

$$\Longrightarrow \quad g_i(x_k + \lambda d_k) < b_i \qquad \text{for } i \in N_a^k$$

i.e. $x_k + \lambda d_k$ satisfies the non-linear constraints.

Similarly, from $d_k \in D(x_k)$, it is easily seen that $x_k + \lambda d_k$ satisfies the linear constrains also. Therefore, we conclude that $x_k + \lambda d_k \in P$ for however small $\lambda > 0$ i.e. d_k is a feasible direction.

Theorem 5. In program (5.1.1), if $X_* \in P$ satisfies the 'Regularity Condition' (C_1), then the necessary and sufficient condition that X_* is a global maximum of φ over P is that

$$\eta'(X_*)\, d_* \leq 0 \qquad \text{for all} \qquad d_* \in D(X_*) \tag{5.1.26}$$

where, $D(X_*)$ is the convex polyhedral cone (5.1.11).

Proof. 'The Condition is Necessary': Suppose that a $d_* \in D(X_*)$ exists, such that $\eta'(X_*)\, d_* > 0$ is satisfied. Since X_* satisfies the regularity condition C_1, therefore, there exists some $\bar{d}_*$ satisfying $\eta_i'(X_*)\,\bar{d}_* < 0$ for $i \in N_\alpha^*$, $a_i \bar{d}_* \leq 0$ for $i \in L_\alpha^*$. Let $d_*(\lambda) = \lambda d_* + \bar{d}_*$. Clearly $d_*(\lambda)$ is feasible for all $\lambda \geq 0$ since $\eta_i'(X_*)\, d_*(\lambda) < 0$ for $i \in N_\alpha^*$, $a_i\, d_*(\lambda) \leq 0$ for $i \in L_\alpha^*$, $d_{*j}(\lambda) \geq 0$ for $j \in J_\alpha^{*+}$ and $d_{*j}(\lambda) \leq 0$ for $j \in J_\alpha^{*-}$ will hold for all $\lambda \geq 0$. Moreover $\eta'(X_*)\, d_*$ being > 0, it is possible for us to choose λ so large that $\eta'(X_*)\, d_*(\lambda) > 0$ will hold. This implies that X_* cannot be a maximum point. Hence the result.

'The Condition is Sufficient': This part follows trivially as a consequence of the definition of a u.f.d. and Theorem 2.

Theorem 6. In program (5.1.1), if $X_* \in P$ satisfies the 'Regularity Condition' (C_1), then the necessary and sufficient condition that X_* is a global maximum of φ over P is that the gradient vector $\eta(X_*)$ at X_* is a non-negative linear combination of the outward-pointing normals in X_*, i.e.

$$\left.\begin{aligned} & X_* \text{ is max} \Leftrightarrow \eta(X_*) = \sum_{i\in N_a^*} u_i \gamma_i(X_*) + \sum_{i\in L_a^*} u_i a_i - \sum_{j\in J_a^{*+}} v_j e_j + \sum_{j\in J_a^{*-}} v_j e_j \\ & \qquad u_i \geqslant 0, v_j \geqslant 0 \quad , \quad i \in N_a^* \cup L_a^* \quad , \quad j \in J_a^{*+} \cup J_a^{*-} \end{aligned}\right\} \tag{5.1.27}$$

Proof. If $\eta(X_*)$ can be written as in (5.1.27) then for any $d_* \in D(X_*)$ we have $\eta'(X_*)\, d_* \leq 0$ so that by Theorem 6, X_* is a global maximum of φ on P. On the other hand if X_* is a global maximum of φ on P, then Theorem 6 shows that $\eta'(X_*)\, d_* \leq 0$ for any $d_* \in D(X_*)$ such that (5.1.27) follows by applying Farkas' Lemma [41, page 164] .

Theorem 7. In program (5.1.1), if the O.F. φ be SPCV and it satisfies the conditions (5.1.17) with $m < \infty$, then the set

$$P_\alpha = \left\{ X \;;\; X \in P,\ \varphi(X) \geqslant \alpha \right\}$$

is bounded for all $\alpha \in R$, provided $K(X_1, X_2) < \infty \;\forall\; X_1, X_2 \in P$.

Proof. If $\alpha > m$, then P_α is evidently null. If $\alpha \leq m$, then $P_\alpha \supset P_m$ and is convex. Suppose P_α is unbounded for some α , so that, for any $X \in P_m$, a vector d can be found such that $X + \lambda d \in P_\alpha$ for all $\lambda > 0$. Take λ_1 so large that the point $X_1 = X + \lambda_1 d \notin P_m$. Hence $\varphi(X_1) < m = \varphi(X)$. Since φ is SPCV, therefore, from definition of a SPCV function for $\varphi(X_1) < \varphi(X)$ we have $(X - X_1)' \nabla_X \varphi(X_1) > 0$ so that $\eta'(X_1)\, d < 0$ holds. Now $X + \lambda d = X_1 + (\lambda - \lambda_1) d$ so that using the strong pseudo-concavity of φ we obtain for $K(X + \lambda d, X_1) > 0$, $K(X + \lambda d, X_1) < \infty$

$$(\lambda - \lambda_1)\, \eta'(X_1)\, d \geqslant K\left(X_1 + (\lambda - \lambda_1) d, X\right)\left[\varphi(X_1 + (\lambda - \lambda_1) d) - \varphi(X_1)\right]$$

i.e. $$\varphi(X + \lambda d) \leq \varphi(X_1) + \frac{(\lambda - \lambda_1)\, \eta'(X_1)\, d}{K(X + \lambda d, X_1)} < \alpha$$

if λ is large enough as $0 < K(x+\lambda d, x_1) < \infty$ and $\eta'(x_1)d < 0$.
Hence $x+\lambda d \in P_\alpha$ for all $\lambda > 0$ cannot hold, i.e. P_α is bounded.

Corollary 1: If $m < \infty$, then for any $Y \in P$, the set $\{x;\, x \in P, \varphi(x) \geq \varphi(Y)\}$ is bounded, provided $K(x,y) < \infty \ \forall\, x \in P$.

Remark 2: It is very important to remark over here that Theorem 7 has been proved by Zoutendijk [185, page 64] for the case when the function φ is CV. Here, however, we have shown that the theorem still holds good even if φ is SPCV which is more general than a CV function. The Theorem does not, however, necessarily hold good when the function φ belongs to the class of PCV (but not SPCV) functions, a class more general than that of SPCV functions. Therefore, when φ is PCV, we have assumed in (5.1.18) that the Theorem 7 holds. This result is of fundamental importance in establishing the result that either the sequence (X_k) of feasible solution X_0, X_1, X_2, ------, generated during the solution of (5.1.1) with the help of 'Method of Feasible Directions' has a cluster point or the function φ is unbounded on P.

Remark 3. Thus from the theory developed and the assumptions made we conclude that the method of feasible directions can be applied to the problem (5.1.1), exactly in the manner described by Zoutendijk [184,185], and the solution so obtained will be the global optimum (maximum).

OUTLINES FOR THE 'METHOD OF FEASIBLE DIRECTIONS' AS APPLIED TO SFV P.P.(SCV) PROGRAM (5.1.1): The problem is directly attacked by starting with an initial feasible solution $X_0 \in P$ assumed to be known (if not then can be obtained by known techniques [18]) such that $\nabla_x \varphi(x_0) \neq 0$, and obtaining a sequence of points $X_1, X_2, \text{------}, X_k \in P$ satisfying $\varphi(x_r) > \varphi(x_{r-1})$ for $r=1,2,\text{------},k$. Having generated the sequence $X_0, X_1, \text{------}, X_k$ of feasible solutions a point $X_{k+1} \in P$ is determined. The first step in calculating X_{k+1} is to determine a direction d_k. One property of d_k is that by moving away from X_k in the direction d_k, the value of the O.F. increases or at least it initially increases. Proceed in the direction d_k from X_k until one of the two events happens; either, (i) $\varphi(X)$ is maximized in this direction or (ii) proceeding further in this direction would go beyond the feasible set. The first point at which either of the two events happens is X_{k+1}. The method of computing X_{k+1} is such that either it is ensured that $\varphi(X_{k+1}) > \varphi(X_k)$ or is concluded that no better point X_{k+1} exists so that X_k is the desired maximum of the problem (5.1.1). The choice of the directions d_k and the length λ_k of the steps taken to move from X_k to X_{k+1} and the choice of the points X_k is such that either the convergence to the maximum m of φ on P is obtained or that it is concluded that φ is unbounded over P.

We now give in brief the procedure for obtaining the direction d_k and the step-length λ_k such that in moving from X_k to $X_{k+1} = X_k + \lambda d_k \in P$ $\varphi(X_{k+1}) > \varphi(X_k)$, φ increases along d_k at maximum possible

rate and the increase in the value of φ at each step is maximum possible. For that we assume that we are at X_0 and we want to find X_1 such that we are interested in computing d_o and λ_o .

DIRECTION FINDING PROBLEM: To find a u.f.d. d_o through $X_0 \in P$ we solve, as given by Zoutendijk [105] , direction finding problem given by (5.1.13) for $k=0$, and call the solution, a b.u.f.d. available at X_0. Any d_o with (d_o, σ) satisfying $(d_o, \sigma) \in D(X_o, \sigma)$ with $\sigma > 0$ will be a u.f.d. since, if $\sigma > 0$, we shall have $\psi_i'(X_o)\, d_o < 0$ for $i \in N_a^o$ and $\eta'(X_o)\, d_o > 0$ and if $\sigma \leq 0$ holds for all d_o with (d_o, σ) satisfying $(d_o, \sigma) \in D(X_o, \sigma)$ we have X_0 as the optimal solution.

In case the set N be a null set, then the direction finding problem reduces to solve (5.1.14) for $k=0$. Here any d_o satisfying $d_o \in D(X_o)$ with $\eta'(X_o)\, d_o > 0$ will be a u.f.d. but the requirement, $\eta'(X_o)\, d_o$ is maximized, will lead us to the b.u.f.d. along which the rate of increase of φ will be maximum. This b.u.f.d. will be among those which satisfy normalization requirement. If for the maximum vector $d_{o\,max}$ we have $\eta'(X_o)\, d_{o\,max} \leq 0$, then the optimality criterion (5.1.26) for X_0 being the global maximum of φ on P is satisfied. If $d_{o\,max} = 0$, then $\eta'(X_o)\, d_{o\,max} = 0$ and this also implies that X_0 is global maximum.

STEP LENGTH FINDING PROBLEM:

We assume that through X_0 we have found a u.f.d , d_o , say. We now move from X_0 along d_0, by making a step of length λ_o ,say, so large that,

(i) none of the constraints is violated by the new trial solution $X_1 = X_0 + \lambda d_0$;

(2) $\varphi(X_0 + \lambda d_0)$ as a function of λ will be maximized, in the direction d_0 , subject to the condition that $X_1 \in P$

For this we define

$$\lambda^* = max\{\lambda ;\ X_0 + \lambda d_0 \in P\} \tag{5.1.28}$$

From (5.1.28) it is now guaranted that the step-length λ_0 of the move must be $\leqslant \lambda^*$ ($\lambda^* = \infty$ is possible if some of the p_i are ∞). Again let λ^{**} be the smallest λ such that

$$\varphi(X_0 + \lambda^{**} d_0) = \max_\lambda \varphi(X_0 + \lambda d_0) = \max_\lambda \psi(\lambda) \tag{5.1.29}$$

Here $0 < \lambda^{**} \leq \infty$ will hold.

Then we choose,

$$\lambda_0 = min(\lambda^*, \lambda^{**}) \tag{5.1.30}$$

If $\lambda_0 = \infty$, and φ is SPCV then by the assumption (5.1.17) for φ , ensures that $m = \infty$ will hold so that φ is unbounded, for if $m < \infty$, then $P_{\varphi(X_0)} = \{X ; X \in P,\ \varphi(X) \geqslant \varphi(X_0)\}$ is bounded, so that $\lambda = \infty$ cannot hold. Again when $\lambda_0 = \infty$ and φ is PCV, then the assumption (5.1.18) ensures that φ is unbounded in this case and if λ is not infinity and an infinite sequence of points (X_k) is generated, then if the sequence has no cluster point then φ will be unbounded and if φ is bounded on P, then the set $P_{\varphi(X_0)}$ is bounded. Since $P_{\varphi(X_0)}$ is closed also, therefore it is compact. The function φ being continuous, and the sequence $(\varphi(X_k))$ being a monotonic increasing (obvious consequence of Lemma 2, Theorem 1

and (5.1.30) sequence imply that $X_k \in P_\alpha$ for all k such that by compactness of P_α we obtain the existence of cluster point of infinite sequence (X_k). To find λ^*, we first find $\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*$ as explained below, and then choose

$$\lambda^* = \min(\lambda_1^*, \lambda_2^*, \lambda_3^*, \lambda_4^*) \tag{5.1.31}$$

To find λ_1^*, we find the largest root of each of the equations

$$g_i(X_0 + \lambda d_0) = b_i \quad \text{for} \quad i \in N \tag{5.1.32}$$

and then take the smallest of the figures so obtained. For each $i \in N$, this for instance can be done by Newton's Method.

$$\lambda_2^* = \min_i \left\{ \frac{b_i - a_i X_0}{a_i d_0} \; ; \; a_i d_0 > 0 \; , i \in L - L_a^0 \right\} \tag{5.1.33}$$

$$\left.\begin{aligned} \lambda_3^* &= \min_j \left\{ \frac{x_{0j}}{-d_{0j}} \; ; \; d_{0j} < 0 \; , \; j \in J - J_a^{0+} \right\} \\ \lambda_4^* &= \min_j \left\{ \frac{p_j - x_{0j}}{d_{0j}} \; ; \; d_{0j} > 0 \; , \; j \in J - J_a^{0-} \right\} \end{aligned}\right] \tag{5.1.34}$$

Thus by repeating the process we obtain the sequence of feasible points $X_0, X_1, X_2, \text{------}, X_k \in P$ satisfying $\varphi(X_r) > \varphi(X_{r-1})$ for r=1,2,------, k, untill, either (i) the optimality condition (5.1.26) is satisfied, or, (ii) there exists a λ_k for $0 < \lambda_k \leq \epsilon$ for $\epsilon > 0$ however small, such that $X_k + \lambda_k d_k \in P$, or (iii) $\varphi(X_k) - \varphi(X_{k-1}) < \delta$ where δ is a small positive number.

Remark: Although the method of feasible directions as described above is a non-finite method yet it is convergent to an optimal solution. However, without some precautionary measures the sequence (X_k)

of points X_k may converge to a non-stationary (and thus non-optimal) point. To prevent this convergence to a non-stationary point, a so called 'Anti-zigzagging Precaution' [185] is used and it is easily established as in Zoutendijk [185], that with this precaution any point of accumulation of the sequence (X_k) is also a stationary point.

Remark 4: In general case the problems of finding λ^{**} and λ^{*} are of the same type. But if the objective function φ be (i) the product of two linear factors, (ii) ratio of a quadratic factor to a linear factor; then the problem of finding λ^{**} (if such a λ^{**} exists) from $\frac{d\psi(\lambda)}{d\lambda} = 0$, which will be respectively a (i) linear and (ii) quadratic equation in λ, can be solved easily. Also if all the constraints be linear then λ^{*} also is easily obtainable.

We now illustrate our procedure by solving the following simple numerical example of quadratic indefinite functional programming.

NUMERICAL EXAMPLE

Maximize $\varphi(x_1, x_2) = (x_1+1)(2x_2+1)$

subject to

$$x_1 + x_2 \leq 2$$

$$x_1 \geq 0$$

$$x_2 \geq 0$$

Now $\eta'(X) = [2x_2+1,\ 2(x_1+1)]$ where $X = (x_1, x_2)'$

Iteration 1. Let $X_0 = (x_{01}, x_{02})' = (0,0)'$ be the initial feasible solution. Then the initial direction finding problem is

Maximize $d_1 + 2d_2$

where d_1, d_2 satisfy

$$d_1 \geq 0$$
$$d_2 \geq 0$$
$$-1 \leq d_1 \leq 1$$
$$-1 \leq d_2 \leq 1$$

and the solution is $d_{(0)} = (d_{01}, d_{02})' = (1,1)'$ and $\eta'(X_0)\, d_{(0)} = 3 > 0$ which show that the direction $(1,1)'$ is the best u.f.d. at $(0,0)'$ and $\varphi(X_0) = 1$.

Now we solve the 'step length finding' problem. For that we first find λ^{**} by solving

$$\frac{d\psi(\lambda)}{d\lambda} = \eta'(\lambda,\lambda)\, d_{(0)} = 0 \qquad \text{where} \quad d_{(0)} = (d_{01}, d_{02})'$$

i.e. $2\lambda + 1 + 2(\lambda+1) = 0 \Rightarrow \lambda = -3/4$

which does not satisfy $0 < \lambda \leq \varepsilon$, therefore, there is no λ^{**} along which the function will increase at maximum rate.

Now we find λ^* from

$$\lambda^* = \frac{2-0-0}{2} \qquad \text{for} \quad x_1 + x_2 < 2 \ \text{at} \ (0,0)'$$
$$= 1$$

Thus
$$\lambda_0 = \min(\lambda^*, \lambda^{**})$$
$$= 1$$

and we get $X_1 = (x_{11}, x_{12})' = (1,1)'$, $\varphi(X_1) = 6$

This completes the first iteration.

Iteration 2. Now we have $X_1 = (1,1)'$. Therefore at this stage the Direction finding problem is

Maximize $3d_1 + 4d_2$

subject to

$$d_1 + d_2 \leq 0$$
$$-1 \leq d_1 \leq 1$$
$$-1 \leq d_2 \leq 1$$

and the solution is $d_{(1)} = (d_{11}, d_{12})' = (-1, 1)'$, and $\eta'(x_1)\, d_{(1)} = 1 > 0$.
showing that the direction $(-1,1)'$ is the best u.f.d. at $(1,1)'$.

To find step-length we have,

(i) for λ^{**}

$$\frac{d\,\psi(\lambda)}{d\lambda} = \eta'(1-\lambda, 1+\lambda)\, d_{(1)} = 0 \qquad \text{where} \quad d_{(1)} = (d_{11}, d_{12})'$$

$$\Rightarrow -2\lambda - 3 + 4 - 2\lambda = 0$$

$$\Rightarrow \lambda = \frac{1}{4}$$ which satisfies $0 < \lambda \leq \varepsilon$

Therefore, $\lambda^{**} = \frac{1}{4}$

Now we find λ^* from

$$\lambda^* = \min \left\{ \frac{1}{-(-1)} \right\}$$
$$= 1$$ (Because we have only $d_{11} < 0$)

Thus we obtain

$$\lambda_1 = \min(\lambda^*, \lambda^{**}) = \min(1, \tfrac{1}{4})$$
$$= \tfrac{1}{4}$$

and we get $X_2 = (x_{21}, x_{22})' = (\frac{3}{4}, \frac{5}{4})'$ and $\varphi(X_2) = 6\frac{1}{8}$

Iteration 3. To start with now we have the feasible solution $X_2 = (\frac{3}{4}, \frac{5}{4})'$. Therefore at this state the direction finding problem is,

Maximize $\frac{7}{2}(d_1 + d_2)$

subject to

$$d_1 + d_2 \leq 0$$
$$-1 \leq d_1 \leq 1$$
$$-1 \leq d_2 \leq 1$$

and the solution is $d_{(2)} = (d_{21}, d_{22})' = (0, 0)'$

such that $\eta'(X_2)\, d_{(2)} = 0$ implies that the optimality condition is satisfied. Hence we have obtained $x_1 = \frac{3}{4}$, $x_2 = \frac{5}{4}$ as the optimal solution to the given problem such that $\varphi(X_{opt}) = 6\frac{1}{8}$.

Remark 5: We see that here we have solved the problem in three iterations. However, the method is non-finite in general.

SECTION - II

SPECIAL TYPE OF NON-LINEAR FRACTIONAL FUNCTIONAL PROGRAMMING:

In this section we are mainly concerned with the problem:

$$\text{Maximize } Z = \frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta} + \left[\frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right]^3 + \cdots + \left[\frac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right]^{2p+1} \quad (5.2.1)$$

subject to

$$\left.\begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, \ldots, m) \\ x_j \geq 0 & (j = 1, 2, \ldots, n) \end{array}\right\} \quad (5.2.2)$$

where p is a non-negative finite integer and the following assumptions hold.

(i) All a_{ij}, b_i, c_j, d_j, α and β are known constants.

(ii) To avoid the questions of attainment of maximum it is assumed that the set, P, of feasible solution given by (5.2.2.), viz.

$$P = \left\{ (x_1, x_2, ----, x_n)' \; ; \; \begin{array}{ll} \sum_{j=1}^{n} a_{ij} x_j \leq b_i & (i = 1, 2, ----, m) \\ x_j \geq 0 & (j = 1, 2, ----, n) \end{array} \right\}$$

is regular, i.e. bounded and non-empty.

(iii) $\sum_{j=1}^{n} d_j x_j + \beta > 0$ for all the feasible solutions.

The O.F. (5.2.1) is proved to be PCM over the set of feasible solutions such that the possibility of the existence of an 'Adjacent Vertex Method' for obtaining the solution of the N.L.F.F.P.P. considered above is ensured. Since the constraint set (5.2.2.) is a convex polyhedral set, therefore, it has a finite number of extreme points. Hence the method developed is finite iteration method.

Theorem 1. If f be a PCM function on a convex set $S \subset R^n$, then the function φ defined by

$$\varphi(X) = a_0 + a_1 f(X) + a_3 [f(X)]^3 + ----- + a_{2p+1} [f(X)]^{2p+1} \qquad (5.2.3)$$

where, a_0 is an arbitrary scalar constant, p is finite and $a_1 > 0, a_3, ---, a_{2p+1}$ are non-negative arbitrary scalar constants, is also PCM.

Proof. Let X_1, X_2 be any two points in S. Let f_1 and f_2 stand for $f(X_1)$ and $f(X_2)$ respectively. Therefore, for $a_0, a_1, a_3, ----, a_{2p+1}$ as specified in the theorem, we have that

$$f_1 \geq f_2 \Rightarrow a_0 + a_1 f_1 + a_3 f_1^3 + ---- + a_{2p+1} f_1^{2p+1} \geq a_0 + a_1 f_2 + a_3 f_2^3 + --- + a_{2p+1} f_2^{2p+1}$$

$$\Rightarrow \varphi(X_1) \geq \varphi(X_2) \qquad (5.2.4)$$

Similarly $\quad f_1 \leqslant f_2 \Rightarrow \varphi(X_1) \leqslant \varphi(X_2)$ (5.2.5)

Now $\quad \nabla_x \varphi(X_2) = \left[a_1 + 3a_3 f_2^2 + \cdots\cdots + (2p+1)\, a_{2p+1} f_2^{2p}\right] \nabla_x f_2$

Therefore $\quad (X_1 - X_2)' \nabla_x \varphi(X_2) = \left[\sum_{s=0}^{p} (2s+1)\, a_{2s+1} f_2^{2s}\right] (X_1 - X_2)' \nabla_x f_2$

Using the strict positivity of a_1 and non-negativity of $a_3, \cdots\cdots, a_{2p+1}$ we obtain from (5.2.6) that,

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant 0 \Rightarrow (X_1 - X_2)' \nabla_x f_2 \geqslant 0 \qquad (5.2.7)$$

and

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leqslant 0 \Rightarrow (X_1 - X_2)' \nabla_x f_2 \leqslant 0 \qquad (5.2.8)$$

Using the pseudo-monotonicity of f over S, we obtain from (5.2.7), (5.2.4) and (5.2.8),(5.2.5) respectively, that

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \geqslant 0 \Longrightarrow \varphi(X_1) \geqslant \varphi(X_2)$$

and

$$(X_1 - X_2)' \nabla_x \varphi(X_2) \leqslant 0 \Longrightarrow \varphi(X_1) \leqslant \varphi(X_2)$$

i.e. φ is PCM over S.

Corollary 1. $\dfrac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta} + \left[\dfrac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right]^3 + \cdots\cdots + \left[\dfrac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}\right]^{2p+1}$

as given in (5.2.1) is PCM over (5.2.2).

The proof of the Corollary 1 follows immediately as a particular case, since $\dfrac{\sum_{j=1}^{n} c_j x_j + \alpha}{\sum_{j=1}^{n} d_j x_j + \beta}$ is PCM over (5.2.2)

Theorem 2. If f be a non-negative PCM function defined on a convex

set $S \subset R^n$, then the function φ defined by

$$\varphi(x) = a_0 + a_1 f(x) + a_2 [f(x)]^2 + \text{-----} + a_p [f(x)]^p \qquad (5.2.9)$$

where, p is a non-negative finite integer, a_0 is an arbitrary scalar constant and $a_1 > 0$, $a_2, a_3, \text{------} a_p$ are non-negative arbitrary scalar constants, is also PCM over S.

Proof of Theorem 2 follows on the lines of the proof provided for Theorem 1.

Remark 1. It is to remark that Theorem 2 is slightly more general than Theorem 1 in the sense that it includes all natural powers of f, however, a strong condition of non-negativity is there on f in Theorem 2 which is not there in Theorem 1.

Algorithm: We now describe the computational algorithm developed to solve the programming problem given by (5.2.1) and (5.2.2).

In (5.2.2) we introduce the slack variables and obtain,

$$\left.\begin{aligned} \sum_{j=1}^{n} a_{ij} x_j + x_{n+i} &= b_i \qquad (i=1,2,\text{---},m) \\ x_{n+i} &\geq 0 \qquad (i=1,2,\text{---},m) \\ x_j &\geq 0 \qquad (j=1,2,\text{---},n) \end{aligned}\right\} \qquad (5.2.3)'$$

We now assume that the constraints in (5.2.3)' are such that the rank of the coefficient matrix in the first m equations is equal to m and that the rank of the coefficient matrix is equal to the rank of the augmented matrix.

As in [16,122] we express the objective function Z and the basic variables in terms of non-basic variables, for which, we first

assume, for the sake of notational convenience, that the variables are so numbered that first m variable $x_1, x_2, \text{------}, x_m$ are the basic variables. Thus from $(5.2.3)'$ we obtain

$$\left.\begin{aligned} x_s &= \gamma_{s0} + \sum_{q=1}^{n} \gamma_{sq}\, z_q \qquad (s = 1, 2, ---, m) \\ \text{where} \qquad z_q &= x_{m+q} \qquad\qquad (q = 1, 2, ---, n) \end{aligned}\right\} \qquad (5.2.4)'$$

and the set of variables z_q is the set of non-basic variables.

From $(5.2.4)'$ we infer that our initial basic feasible solution is given by

$$\left.\begin{aligned} x_s &= \gamma_{s0} \qquad (s = 1, 2, ----, m) \\ z_q &= 0 \qquad (q = 1, 2, ----, n) \end{aligned}\right\} \qquad (5.2.5)'$$

(5.2.1) in conjunction with $(5.2.4)'$ yields

$$Z = \frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j} + \left[\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \alpha_j z_j}\right]^3 + \text{------} + \left[\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}\right]^{2p+1} \qquad (5.2.6)'$$

i.e. the objective function Z has been expressed in terms of non-basic variables z_j .

We now develop the main algorithm.

Step I. Selection of Incoming Variable and Optimality Criterion:

We Consider the partial derivatives of Z with respect to z_j and obtain, for $j = 1, 2, ----, n$

$$\frac{\partial Z}{\partial z_j} = \left[1 + 3\left(\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}\right)^2 + \cdots + (2p+1)\left(\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}\right)^{2p}\right] \times$$

$$\frac{(\beta_0 \alpha_j - \alpha_0 \beta_j) + \sum_{k=1}^{n} (\beta_k \alpha_j - \alpha_k \beta_j)\, z_k}{\left(\beta_0 + \sum_{j=1}^{n} \beta_j z_j\right)^2} \qquad (5.2.7)'$$

Let $T = 1 + 3\left(\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}\right)^2 + \cdots + (2p+1)\left(\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}\right)^{2p}$

$$\Delta_j = \beta_0 \alpha_j - \beta_j \alpha_0 \qquad \delta_{kj} = \beta_k \alpha_j - \alpha_k \beta_j$$

$$N_j = \Delta_j + \sum_{\substack{k=1 \\ k \neq j}}^{n} \delta_{kj} z_k \qquad D_j = \beta_0 + \sum_{j=1}^{n} \beta_j z_j$$

Therefore, from $(5.2.7)'$ we have for $j = 1, 2, \ldots, n$

$$\frac{\partial Z}{\partial z_j} = \frac{T\left(\Delta_j + \sum_{k=1, k \neq j}^{n} \delta_{kj} z_k\right)}{\left(\beta_0 + \sum_{j=1}^{n} \beta_j z_j\right)^2} = \frac{T N_j}{D_j^2} \qquad (5.2.8)'$$

If at the initial basic feasible solution $(5.2.5)'$, $\left(\frac{\partial Z}{\partial z_j}\right)^0$ denotes the value of $\frac{\partial Z}{\partial z_j}$ for $j = 1, 2, \ldots, n$, then we have,

$$\left(\frac{\partial Z}{\partial z_j}\right)^0 = \frac{\left[1 + 3\left(\frac{\alpha_0}{\beta_0}\right)^2 + 5\left(\frac{\alpha_0}{\beta_0}\right)^4 + \cdots + (2p+1)\left(\frac{\alpha_0}{\beta_0}\right)^{2p}\right]\Delta_j}{\beta_0^2} \qquad (5.2.9)'$$

We now make two important observations over here.

1. In $(5.2.8)'$, in the numerator of $\frac{\partial Z}{\partial z_j}$, T containts only even powers of $\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}$, N_j is independent of z_j.

2. Sign of $\left(\frac{\partial Z}{\partial z_j}\right)^0$ in (5.2.9)$'$ is that of Δ_j and vice versa.

(a) Optimality Criterion:

In (5.2.9)$'$ if $\left(\frac{\partial Z}{\partial z_j}\right)^0 \leqslant 0$ i.e. if $\Delta_j \leqslant 0$ then at the initial basic feasible solution (5.2.5)$'$, Z is a non-increasing function of z_j and therefore a small increase in the non-basic variable z_j with other non-basic variables held at the zero level will not increase Z, where Z is given by (5.2.6)$'$.

Therefore if,

$$\Delta_j \leqslant 0 \qquad \text{for all} \qquad j = 1, 2, \text{---}, n \tag{5.2.10}$$

then we have (5.2.5)$'$ as the l cal maximum and hence the global maximum is obtained.

If $\left(\frac{\partial Z}{\partial z_j}\right)^0 > 0$ i.e. $\Delta_j > 0$ for some j=1,2,------,n; this implies that at the initial basic feasible solution (5.2.5)$'$, Z is an increasing function of z_j and therefore a small increase in z_j will increase Z and, therefore, in this case (5.2.5)$'$ is not an optimal solution, and hence it is possible to go on increasing z_j untill we have to stop to avoid,

(1) Making one of the variables negative. (5.2.11)

(2) $\frac{\partial Z}{\partial z_j}$ vanishes and is about to become negative. (5.2.12)

In the problem considered above, we observe from (5.2.8)$'$ that (5.2.11) is the only possibility over here and (5.2.12) cannot happen

because in the numerator of $\frac{\partial Z}{\partial z_j}$ in (5.2.8) ; T is always positive since it contains only even powers of $\frac{\alpha_0 + \sum_{j=1}^{n} \alpha_j z_j}{\beta_0 + \sum_{j=1}^{n} \beta_j z_j}$ and N_j is independent of z_j , and in the denominator D_j^2 also is always positive, therefore, any amount of increase in z_j will not effect the sign of $\frac{\partial Z}{\partial z_j}$.

Thus any z_j for which $\Delta_j > 0$ can be the non-basic variable which is to become the basic variable at the next iteration. We shall call such a variable to be the 'Incoming Variable'. From computational point of view it is always profitable to choose that z_j to be incoming variable which satisfies

$$\max_j \left\{ \Delta_j \; ; \; \Delta_j > 0 \right\}$$

i.e. $$\max_j \left\{ \left(\frac{\partial Z}{\partial z_j}\right)^0 ; \left(\frac{\partial Z}{\partial z_j}\right)^0 > 0 \right\}$$

Selection of Outgoing Vector: When $\left(\frac{\partial Z}{\partial z_j}\right)^0 > 0$ for some $j=1,2,\text{------},n$; We have seen that (5.2.11) is the only possibility. Therefore, we can change the basis as in [16,122] by making some basic variable x_k as non-basic and replacing it by the incoming, variable z_j chosen according to the criterion given above. The variable x_k however is not arbitrarily chosen, it is selected from (5.2.4) by choosing for k that value of s for which

$$\frac{\gamma_{k0}}{|\gamma_{kj}|} = \min_s \frac{\gamma_{s0}}{|\gamma_{sj}|} \quad \text{, where } \gamma_{sj} < 0 . \qquad (5.1.13)$$

Since we have assumed the set P to be regular therefore, one of γ_{sj} (s=1,2,------,m) will surely be negative. We shall call the variable x_k "Outgoing Variable".

Tableau Transformation: Having chosen the Incoming and Outgoing Variables we are now ready to transform the tableau. Since z_j is the incoming variable and x_k is the outgoing variable, therefore we shall use the equation

$$x_k = \gamma_{k0} + \sum_{q=1}^{n} \gamma_{kq} z_q \tag{5.2.14}$$

obtained from (5.2.4)$'$ for $s = k$ to substitute for z_j in terms of the new non-basic variable x_k and other non-basic variables throughout the constraints and also in the objective function Z.

From (5.2.14) we have,

$$z_j = \frac{x_k}{\gamma_{kj}} - \frac{\gamma_{k0}}{\gamma_{kj}} - \sum_{\substack{q=1 \\ q \neq j}}^{n} \frac{\gamma_{kq}}{\gamma_{kj}} z_q \tag{5.2.15}$$

Making use of (5.2.15) to eliminate z_j in (5.2.4)$'$ and (5.2.6)$'$ we obtain

$$x_s = \gamma_{s0} + \gamma_{sj} z_j + \sum_{\substack{q=1 \\ q \neq j}}^{n} \gamma_{sq} z_q$$

$$= \gamma_{s0} + \gamma_{sj} \left[\frac{x_k}{\gamma_{kj}} - \frac{\gamma_{k0}}{\gamma_{kj}} - \sum_{\substack{q=1 \\ q \neq j}}^{n} \frac{\gamma_{kq}}{\gamma_{kj}} z_q \right] + \sum_{\substack{q=1 \\ q \neq j}}^{n} \gamma_{sq} z_q$$

$$= \left(\gamma_{s0} - \gamma_{k0} \frac{\gamma_{sj}}{\gamma_{kj}} \right) + \frac{\gamma_{sj}}{\gamma_{kj}} x_k + \sum_{\substack{q=1 \\ q \neq j}}^{n} \left[\gamma_{sq} - \gamma_{kq} \frac{\gamma_{sj}}{\gamma_{kj}} \right] z_q$$

$$= \bar{\gamma}_{s0} + \bar{\gamma}_{sj} x_k + \sum_{\substack{q=1 \\ q \neq j}}^{n} \bar{\gamma}_{sq} z_q \tag{5.2.16}$$

where

$$\left.\begin{aligned} \bar{\gamma}_{so} &= \gamma_{so} - \gamma_{ko}\frac{\gamma_{sj}}{\gamma_{kj}} \\ \bar{\gamma}_{sq} &= \gamma_{sq} - \gamma_{kq}\frac{\gamma_{sj}}{\gamma_{kj}} \qquad q = 1, 2, ---, n \quad q \neq j \\ \bar{\gamma}_{sj} &= \frac{\gamma_{sj}}{\gamma_{kj}} \end{aligned}\right\} \qquad (5.2.17)$$

and similarly

$$Z = \sum_{v=1}^{p} \left[\frac{\alpha_o + \sum\limits_{\substack{q=1 \\ q \neq j}}^{n} \alpha_q z_q + \alpha_j z_j}{\beta_o + \sum\limits_{\substack{q=1 \\ q \neq j}}^{n} \beta_q z_q + \beta_j z_j} \right]^{2v+1}$$

transforms to

$$\bar{Z} = \sum_{v=1}^{p} \left[\frac{\bar{\alpha}_o + \sum\limits_{\substack{q=1 \\ q \neq j}}^{n} \bar{\alpha}_q z_q + \bar{\alpha}_j x_k}{\bar{\beta}_o + \sum\limits_{\substack{q=1 \\ q \neq j}}^{n} \bar{\beta}_q z_q + \bar{\beta}_j x_k} \right]^{2v+1} \qquad (5.2.18)$$

where

$$\left.\begin{aligned} \bar{\alpha}_o &= \alpha_o - \frac{\gamma_{ko}}{\gamma_{kj}}\alpha_j \\ \bar{\alpha}_q &= \alpha_q - \frac{\gamma_{kq}}{\gamma_{kj}}\alpha_j \qquad q = 1, 2, ----, n \quad q \neq j \\ \bar{\alpha}_j &= \frac{\alpha_j}{\gamma_{kj}} \end{aligned}\right\} \qquad (5.2.19)$$

$$\left.\begin{aligned} \overline{\beta}_0 &= \beta_0 - \frac{\gamma_{k0}}{\gamma_{kj}} \beta_j \\ \overline{\beta}_v &= \beta_v - \frac{\gamma_{kv}}{\gamma_{kj}} \beta_j \qquad v = 1, 2, \ldots, n,\ v \neq j \\ \overline{\beta}_j &= \frac{\beta_j}{\gamma_{kj}} \end{aligned}\right\} \tag{5.2.20}$$

Also $\overline{Z} > Z$.

We now repeat the process with the new objective function $\overline{Z}$ and the new basic and non-basic variables and continue it till the optimality criterion similar to as established in (5.2.10) is satisfied at some stage. This will lead us to the optimal solution.

Convergence of the Method: We remark that during the process of computing the optimal solution starting with an initial basic feasible solution we are always moving to another basic feasible solution only, with an improved value of the O.F. Since our constraint set is nothing but a convex polyhedral set, therefore, the number of extreme points is finite. Hence our process of finding the maximum of the N.L.F.F.P. considered here will terminate in a finite number of steps. Further the nature of the function being such that every local optimum over P is global, therefore, with the help of the method described above we obtain the global maximum in a finite number of steps.

NUMERICAL EXAMPLE

We now illustrate our method by considering a numerical example. We assume that p = 2 in (5.2.1) and consider:

$$\text{Maximize } Z = \frac{3x_1+5x_2+2}{2x_1+3x_2+4} + \left[\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right]^3 + \left[\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right]^5$$

subject to

$$3x_1 + 5x_2 \leq 15$$

$$5x_1 + 2x_2 \leq 10$$

$$x_1, x_2 \geq 0$$

Introducing slack variables we get the problem as:

$$\text{Maximize } Z = \frac{3x_1+5x_2+2}{2x_1+3x_2+4} + \left[\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right]^3 + \left[\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right]^5 \tag{5.2.21}$$

subject to

$$\left.\begin{aligned} 3x_1 + 5x_2 + x_3 &= 15 \\ 5x_1 + 2x_2 + x_4 &= 10 \\ x_j \geq 0 \quad & j = 1, 2, 3, 4 \end{aligned}\right\} \tag{5.2.22}$$

Treating x_3 and x_4 as basic variables we express in the constraints x_3 and x_4 in terms x_1 and x_2 which are non-basic variables.

Thus,

$$\left.\begin{aligned} x_3 &= 15 - 3x_1 - 5x_2 \\ x_4 &= 10 - 5x_1 - 2x_2 \end{aligned}\right\} \tag{5.2.23}$$

Since Z is already in terms of x_1 and x_2 therefore, we directly find the initial basic feasible solution and the value of Z at this initial basic feasible solution.

Initial basic feasible solution is

$$x_3 = 15 \qquad \text{and} \qquad x_1 = 0$$

$$x_4 = 10 \qquad\qquad\qquad x_2 = 0$$

And

$$Z = \cdot 656$$

Differentiating Z partially with respect to x_1 and x_2 be obtain

$$\frac{\partial Z}{\partial x_1} = \left[1 + 3\left(\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right)^2 + 5\left(\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right)^4\right] \frac{8-x_2}{(2x_1+3x_2+4)^2}$$

$$\frac{\partial Z}{\partial x_2} = \left[1 + 3\left(\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right)^2 + 5\left(\frac{3x_1+5x_2+2}{2x_1+3x_2+4}\right)^4\right] \frac{x_1+14}{(2x_1+3x_2+4)^2}$$

Therefore,

$$\left(\frac{\partial Z}{\partial x_1}\right)^0 = \frac{33}{32}, \quad \left(\frac{\partial Z}{\partial x_2}\right)^0 = \frac{231}{128}$$

We see here that $\left(\frac{\partial Z}{\partial x_1}\right)^0 > 0$ and $\left(\frac{\partial Z}{\partial x_2}\right)^0 > 0$. Therefore, it is possible to increase Z by making any of x_1, x_2, a basic variable. We choose x_2 to enter the basic set since

$$\left(\frac{\partial Z}{\partial x_2}\right)^0 = \max_{j=1,2}\left[\left(\frac{\partial Z}{\partial x_j}\right)^0, \left(\frac{\partial Z}{\partial x_j}\right)^0 > 0\right]$$

To choose the outgoing variable we compute the ratios

$$\frac{\gamma_{s0}}{|\gamma_{sj}|} \quad \text{for} \quad \gamma_{sj} < 0 \; , \; s = 3,4 \, , \; j = 2$$

and choose the minimum of them.

From (5.2.4) and (5.2.23) we have

$$\frac{\gamma_{30}}{|\gamma_{32}|} = 3 \, , \; \frac{\gamma_{40}}{|\gamma_{42}|} = 5 \quad \text{where} \quad \gamma_{30} = 15 \; , \; \gamma_{40} = 10$$

$$|\gamma_{32}| = 5 \; , \; |\gamma_{42}| = 2$$

$$\underset{s}{\text{Min}} \; \frac{\gamma_{s0}}{|\gamma_{sj}|} = 3$$

Therefore, x_3 is the outgoing variable, i.e. the variable to become non-basic. Thus it is profitable to increase x_2 upto 3 only and the basic variable x_3 is seen to become non-basic and reduces to zero level.

Computing now, with the help of transformation (5.2.19) and (5.2.20) the new basic variables and the new expression for the objective function in terms of new non basic variables x_1, x_3 we obtain,

$$x_2 = 3 - \frac{3}{5} x_1 - \frac{1}{5} x_3$$

$$x_4 = 4 - \frac{19}{5} x_1 + \frac{2}{5} x_3$$

$$Z = \frac{17-x_3}{\frac{1}{5}x_1 - \frac{3}{5}x_3+13} + \left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^3 + \left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^5$$

Thus we have the new basic feasible solution as,

$$\hat{x}_2 = 3 \qquad \hat{x}_1 = 0$$

and

$$\hat{x}_4 = 4 \qquad \hat{x}_3 = 0$$

and the new value of the objective function as,

$$\hat{Z} = 7.365$$

we now find

$$\frac{\partial Z}{\partial x_1} = \left[1+3\left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^2 + 5\left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^4\right]\left[\frac{-\frac{1}{5}(17-x_3)}{\left(\frac{1}{5}x_1-\frac{3}{5}x_3+13\right)^2}\right]$$

$$\frac{\partial Z}{\partial x_3} = \left[1+3\left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^2 + 5\left(\frac{17-x_3}{\frac{1}{5}x_1-\frac{3}{5}x_3+13}\right)^4\right]\left[\frac{-\frac{1}{5}(x_1+14)}{\left(\frac{1}{5}x_1-\frac{3}{5}x_3+13\right)^2}\right]$$

Therefore,

$$\left(\frac{\partial Z}{\partial x_1}\right)^0 = -\text{ve quantity}.$$

$$\left(\frac{\partial Z}{\partial x_2}\right)^0 = -\text{ve quantity}.$$

This implies we have arrived at the optimal basic feasible solution

$$x_2 = 3 \qquad x_1 = 0$$

$$x_4 = 4 \qquad x_3 = 0$$

With the value of objective function $Z = 7.365$.

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